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الجامعة الأردنيـــــة
كلية الدراسات العليا

التاريـــخ: ١٤ / ٣ / ٢٠١١م

نموذج رقم (١٨)
اقرار والتزام بالمعايير الأخلاقية والأمانة العلمية وقوانين الجامعة الأردنية وأنظمتها وتعليماتها لطلبة الماجستير

أنا الطالب: هدى علي حميدان الوحشا الرقم الجامعي: (٨٠٨٠١٥١)

تخصص: الرياضيات الكليــــــة: العلوم

عنوان الرسالة: Pitman Closeness Comparisons of Ordered Data

اعلن بأنني قد التزمت بقوانين الجامعة الأردنية وأنظمتها وتعليماتها وقراراتها السارية المفعول المتعلقة باعداد رسائل الماجستير عندما قمت شخصيا" باعداد رسالتي وذلك بما ينسجم مع الأمانة العلمية وكافة المعايير الأخلاقية المتعارف عليها في كتابة الرسائل العلمية. كما أنني أعلن بأن رسالتي هذه غير منقولة أو مستلة من رسائل أو كتب أو أبحاث أو أي منشورات علمية تم نشرها أو تخزينها في أي وسيلة اعلامية، وتأسيسا" على ما تقدم فانني أتحمل المسؤولية بأنواعها كافة فيما لو تبين غير ذلك بما فيه حق مجلس العمداء في الجامعة الأردنية بالغاء قرار منحي الدرجة العلمية التي حصلت عليها وسحب شهادة التخرج مني بعد صدورها دون أن يكون لي أي حق في التظلم أو الاعتراض أو الطعن بأي صورة كانت في القرار الصادر عن مجلس العمداء بهذا الصدد.

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PITMAN CLOSENESS COMPARISONS OF ORDERED DATA

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This Thesis (PITMAN CLOSENESS COMPARISONS OF ORDERED DATA) was successfully defended and approved on 6/3/2011

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DEDICATION

This thesis is respectfully dedicated to my parents, brothers, sisters, and to all my friends for their help and support.

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Table of Contents

List of Tables

PITMAN CLOSENESS COMPARISONS OF ORDERED DATA

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Supervisor
Dr. Mohammad Z. Raqab, Prof.

Co-Supervisor
Dr. Ahmad Al-Zghoul

ABSTRACT

The Pitman closeness of one-sequence and two-sequence record statistics to the population quantiles of a location scale family of distributions have been recently studied by Ahmadi and Balakrishnan (2009) and Raqab and Ahmadi (2010) respectively. They showed that, for symmetric distributions, the Pitman closeness probabilities of record statistics to the median are distribution-free.

In this thesis, we generalize the work of Ahmadi and Balakrishnan (2009) and Raqab and Ahmadi (2010) to k-record statistics. The Pitman closeness probabilities of one-sequence and two-sequence k-record statistics to the population quantiles are derived. Also, we show that, as in usual record statistics, the Pitman closeness probablilities of k-record statistics to the median, for symmetric distributions, are distribution free. Examples for uniform and exponential distributions are given.

CHAPTER ONE

INTRODUCTION

In this thesis we are interested in the determination of the closest record statistic among those from one sequence or two sequences of random variables, to a specific population quantile. This goal leads to consider the probability of nearness with which one record statistic will be closer to a quantile than another record statistic from the same sequence or from another independent sequence with the same distribution. This concept is known as Pitman's measure of closeness.

The Pitman closeness (PC) was introduced by Pitman (1937) in the context of estimation and this criterion is based on the probabilities of closeness of competing statistics in relation to an unknown parameter. This concept has been studied in great detail in literature; Rao (1981) has investigated this criterion for estimators of the parameters of normal distribution where he critically examines minimum mean square error as a criterion for estimation. Keating and Mason (1985) present some practical situations where the PC criterion is more relevant than minimum mean square error criterion. Mason, et al. (1990) gave a method for the comparison of two linear forms of a common random vector under the criterion of PC measure. Raqab (2007) used the PC measure to compare between different types of prediction for the future order statistics.

Recently, Balakrishnan, et al. (2008) examined the closeness of the sample median to the population median among all order statistics in terms of PC criterion. Balakrishnan, et al. (2009a) examined the PC of order statistics to population quantiles. Balakrishnan, et al. (2009b) compared the best linear unbiased predictor and best linear invariant predictor of the censored order statistics from exponential distribution in the one-sample and order

statistics from a future sample in the two-sample case, in terms of PC criterion. Further, Ahmadi and Balakrishnan (2009) examined the PC of record statistics to population quantiles. Raqab (2010) derived expressions of the PC measure of ordered statistics from two sequences to population quantiles. Raqab and Ahmadi (2010) derived expressions of the PC measure of record statistics from two sequences to population quantiles.

Definition (Pitman's measure of closeness)

If T_1 and T_2 are two estimators of a common parameter θ, then Pitman's measure of closeness of T_1 relative to T_2 is

$$P\left(\left|T_1-\theta\right|<\left|T_2-\theta\right|\right) \qquad \forall\theta\in\Theta.$$

Definition (Pitman-closer)

If T_1 and T_2 are two estimators of a common parameter θ, then T_1 is a Pitman-closer to the parameter θ than T_2 if

$$P\left(\left|T_1-\theta\right|<\left|T_2-\theta\right|\right)\geq\frac{1}{2} \qquad \forall\theta\in\Theta,$$

with strict inequality for at least one θ.

Definition (Pitman-closest)

Let D be a nonempty set of estimators of a common parameter θ, then T* is Pitman-closest to the parameter θ among the estimators in D provided for every $T\in D$, such that $T^*\neq T$

$$P\left(\left|T^*-\theta\right|<\left|T-\theta\right|\right)\geq\frac{1}{2} \qquad \forall\theta\in\Theta,$$

with strict inequality for at least one θ.

1.1 Record Statistics

Let $\{X_i; i \geq 1\}$ be an infinite sequence of independent and identically distributed *(iid)* random variables, each distributed according to a continuous cumulative distribution function *(cdf)* *F*and probability density function *(pdf) f.* An observation X_i is called an upper record if its value exceed that all previous observations. Thus X_i is an upper record if $X_i > X_j$ for every $j < i$. An analogous definition deals with lower record values. In this thesis we are interested in upper records (or simply record statistics). So, by definition $X_{U(1)} = X_1$ is a record statistic.

Now, let $X_{U(n)}$ be the *n*th record statistic corresponding to an *iid* sequence of random variables with continuous *cdf F* and *pdf f*. Then the *pdf* of $X_{U(n)}$ is given by:

$$f_n(x) = \frac{[H(x)]^{n-1}}{(n-1)!} f(x), \quad -\infty < x < \infty \quad , n = 1,2,3,\cdots,$$

The joint *pdf* of $X_{U(m)}$ and $X_{U(n)}$ where $1 \leq m < n$ is given by

$$f_{m,n}(x, y) = \frac{[H(x)]^{m-1}}{(m-1)!} \frac{[H(y) - H(x)]^{n-m-1}}{(n-m-1)!} h(x) f(y), \quad -\infty < x < y < \infty$$

where $H(x) = -\log[1 - F(x)]$ is the cumulative hazard function and $h(x) = \dfrac{f(x)}{1 - F(x)}$ is the corresponding hazard rate of *X*.

The joint *pdf* of $X_{U(1)}, X_{U(2)}, \cdots, X_{U(n)}$ (the first n record statistics) is

$$f_{1,2,\cdots,n}(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n-1} h(x_i) f(x_n), \qquad -\infty < x_1 < x_2 < \cdots < x_n < \infty .$$

For more details one may refer to Arnold, et al. (1992) and Ahsanullah and Raqab (2006).

Interest on record values has increased since Chandler first formulated the theory of record statistics in 1952 (Chandler, 1952). Record values arise in a wide variety of practical situations such as industrial stress testing, meteorological analysis, hydrology, seismology, sporting, athletic events, oil and mining surveys. For example, record values are used in shock models, and they are closely connected with the occurrence times of a corresponding non-homogeneous Poisson process (see Kamps, 1994). Another example, Glick (1978) studies the destructive testing of wooden beams in which the first beam is subjected to increasing stress until it breaks, thereafter, beams are subjected to increasing stress until they break or the stress reaches the stress needed to break the previous broken beam. In this way, a beam breaks only if its strength is a record value, with the total.

1.2 K-Record Statistics

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables with a continuous *cdf F* and *pdf f*. Denote by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ the order statistics of $X_1, X_2, \cdots, X_n$. For a fixed integer k ≥ 1, the corresponding *k*-record statistics $\{X_{U(n,k)}; n \geq 2\}$ are defined as (see Arnold, et al. (1992)):

Let $T_{1,k} = k$, $X_{U(1,k)} = X_{1:k}$

For n ≥ 2, let

$$T_{n,k} = \min\{j : j > T_{n-1,k} \ , X_j > X_{T_{n-1,k} - k+1 : T_{n-1,k}}\} \ ,$$

and define $X_{U(n,k)} = X_{T_{n,k} - k+1 : T_{n,k}}$, that is $X_{U(n,k)}$ is a *k*-record statistic if it is the *k*th greatest value of a sample of size n. For k = 1, note that the usual record statistics are recovered. Further, when k = 2, $X_{U(1,2)}$ is the first value of the 2-records if it is a minimum

out of a sample of size 2. These sequences of k-record statistics were introduced by Dziubdziela and Kopocinski (1976). If $X_{U(n,k)}$ is a *k*-record statistic from this sequence then, with k, n $\geq$ 1, its marginal *pdf* is given by

$$f_{n,k}(x) = \frac{k^n [H(x)]^{n-1}}{(n-1)!} [1 - F(x)]^{k-1} f(x), \qquad -\infty < x < \infty \;,$$

and its *cdf* is

$$F_{n,k}(x) = \int_{-\infty}^{x} f_{n,k}(t)\, dt = 1 - [1 - F(x)]^k \sum_{j=0}^{n-1} \frac{k^j [H(x)]^j}{j!}, \quad -\infty < x < \infty \;.$$

Also, if $X_{U(m,k)}$ and $Y_{U(n,k)}$ are two k-record statistics ($X_{U(m,k)} < Y_{U(n,k)}$) taken from the same infinite sequence with $1 \leq m < n$, then their joint *pdf* is given by

$$f_{m,n,k}(x,y) = \frac{k^n [H(x)]^{m-1} h(x)}{(m-1)!} \frac{[H(y) - H(x)]^{n-m-1}}{(n-m-1)!} [1 - F(y)]^{k-1} f(y), \qquad -\infty < x < y < \infty$$

For more details one may refer to Arnold, et al. (1992) and Ahsanullah and Raqab (2006).

1.3 Preliminary Details and Definitions

In this section, we recall some definitions and useful identities used frequently throughout this thesis.

Let $\zeta(p)$ be the *p*th quantile of a distribution *F*, that is,

$$F(\zeta(p)) = P(X \leq \zeta(p)) = p, \quad p \in (0, 1)$$

or equivalently

$$\zeta(p) = F^{-1}(p) = \sup_{p \in (0,1)} \{x : F(x) \leq p\} .$$

Further, assume $\{X_i; i \geq 1\}$ is an infinite sequence of *iid* random variables from a continuous *cdf* F and *pdf* f and $X_{U(1)} < X_{U(2)} < \cdots < X_{U(r)}$ denote the corresponding first r record statistics. Suppose *F(x)* belongs to the location-scale family of distribution. That is,

$$F(x) = G\left(\frac{x-\mu}{\sigma}\right), \qquad f(x) = \frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right), \qquad \mu \in R, \sigma > 0,$$

where G and g are the corresponding standard forms (with μ =0 and σ =1). Then, it is evident that, $\zeta(p) = \mu + \sigma\zeta^*(p)$ and $X_{U(r)} = \mu + \sigma X^*_{U(r)}$,

where $\zeta^*(p)$ is the pth quantile of G and $X^*_{U(r)}$ is the rth record statistic from the distribution G. Therefore, it suffices to study the PC of record statistics to population quantile $\zeta(p)$ for the standard *G(x)* in one-sample and two-sample problems. In the following lemma, we deal with one-sample problem.

Lemma 1.1

Suppose that $X_{U(r)}$ $, r \in \{1, 2, 3, \cdots\}$ is the Pitman-closest record statistic to $\zeta(p)$ than $X_{U(s)}$ $, s \in \{1, 2, 3, \cdots\} \backslash \{r\}$. Then, $X^*_{U(r)}$ is the Pitman-closest to $\zeta^*(p)$ than $X^*_{U(s)}$.

Proof:

$$P\left(\left|X_{U(r)} - \zeta(p)\right| < \left|X_{U(s)} - \zeta(p)\right|\right)$$

$$= P\left(\left|(\mu + \sigma X^*_{U(r)}) - (\mu + \sigma\zeta^*(p)\right| < \left|(\mu + \sigma X^*_{U(r)}) - (\mu + \sigma\zeta^*(p)\right|\right)$$

$$= P\left(\left|X^*_{U(r)} - \zeta^*(p)\right| < \left|X^*_{U(s)} - \zeta^*(p)\right|\right) \geq \frac{1}{2}$$

which establishes the required result. In a similar manner, the proof of two-sample problem can be established.

It is known that $U_r \overset{d}{=} F(X_{U(r)})$, where U_r is the rth record statistic from an infinite sequence of *iid* standard uniform $U(0,1)$ random variables, and $\overset{d}{=}$ denotes identical in distribution, (see Raqab and Ahmadi (2010)).

In this context, the *cdf* of the rth record statistic from standard uniform $U(0,1)$ can be expressed as

$$G_r(u) = 1 - \sum_{i=0}^{r-1} \frac{[-\log(1-u)]^i}{i!}(1-u), \quad 0 < u < 1$$

its respective *pdf* is

$$g_r(x) = \frac{[-\log(1-u)]^{r-1}}{(r-1)!}, \qquad 0 < u < 1.$$

Now, let $\Gamma(\alpha)$ denote the complete gamma function for a positive constant α,

$$\Gamma(\alpha,\beta;x) = \int_x^\infty u^{\alpha-1} e^{\frac{-u}{\beta}}\, du, \quad x > 0,\ \alpha,\beta > 0$$

the incomplete gamma function, and

$$I(\alpha,\beta;x) = \frac{\Gamma(\alpha,\beta;x)}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_x^\infty u^{\alpha-1} e^{\frac{-u}{\beta}}\, du,$$

the incomplete gamma ratio. The following identity:

$$I(n,\beta;x) = \sum_{j=0}^{n-1} \frac{\left(\frac{x}{\beta}\right)^j e^{\frac{-x}{\beta}}}{j!}, \quad n \geq 1, \tag{1.1}$$

represents the relationship between the incomplete gamma ratio and the Poisson sum of probabilities which will be quite helpful in simplifying some arguments in the proofs of the results established throughout this thesis.

Further, the expression "symmetric distribution" will arise frequently in the subsequent sections of this thesis, so it is worth to mention the definition of symmetric distribution which is given as follows:

The distribution *F(x)* is symmetric about α if the probability that $X \geq \alpha + u$ is the same as the probability that $X \leq \alpha - u$ for every value of *u*. That is,

$$1 - F(\alpha + u) = F(\alpha - u).$$

For special case of $\alpha = 0$, *F(x)* is symmetric about 0 if $1 - F(u) = F(-u), \quad \forall u \in R$.

CHAPTER TWO

PITMAN CLOSENESS OF ONE-SEQUENCE RECORD STATISTICS

In this chapter we reviewed the derivation of general expressions for PC of record statistics to the population quantiles of a location-scale family of distributions. For symmetric distributions, Pitman closeness probabilities of record statistics to the median are shown to be distribution-free. Examples including uniform and exponential distributions are discussed. For more details see Balakrishnan, et al. (2009) and Ahmadi and Balakrishnan (2009).

2.1 PC of Record Statistics

Let us denote the PC probability of any two record statistics $X_{U(r)}$ and $X_{U(i)}$ to a specific population quantile $\zeta(p)$ by

$$\pi(r,i;p) = \mathrm{P}\,(|\, X_{U(r)} - \zeta(p)\,| < |\, X_{U(i)} - \zeta(p)\,|).$$

In the following theorem, we derive a general explicit expression for the probability $\pi(r,i;p)$.

Theorem 2.1

Let $\{X_i, i \geq 1\}$ be a sequence of *iid* random variables with continuous *cdf F(x)* and *pdf f(x)*. Then with $p \in (0,1)$ and $q = 1 - p$, we have for $1 \leq r < i$

$$\pi(r,i;p) = q\sum_{j=0}^{r-1}\frac{(-\log q)^j}{j!} + \sum_{j=0}^{i-r-1}\frac{1}{(r-1)!\,j!}\int_{-\infty}^{\zeta(p)} m_{j,r}(x)\,dx\,, \tag{2.1}$$

and for $1 \leq i < r$

$$\pi(r,i;p) = 1 - q\sum_{j=0}^{i-1}\frac{(-\log q)^j}{j!} - \sum_{j=0}^{r-i-1}\frac{1}{(i-1)!\,j!}\int_{-\infty}^{\zeta(p)} m_{j,i}(x)dx\,, \tag{2.2}$$

where

$$m_{j,r}(x) = [H(2\zeta(p)-x) - H(x)]^j[1 - F(2\zeta(p)-x)][H(x)]^{r-1}h(x)\,. \tag{2.3}$$

Proof: For $1 \leq r < i$

$$\begin{aligned}
\pi(r,i;p) &= \mathrm{P}(|X_{U(r)} - \zeta(p)| < |X_{U(i)} - \zeta(p)|) \\
&= \mathrm{P}([X_{U(r)} - \zeta(p)]^2 < [X_{U(i)} - \zeta(p)]^2) \\
&= \mathrm{P}(X_{U(r)}^2 - 2\zeta(p)X_{U(r)} + \zeta^2(p) < X_{U(i)}^2 - 2\zeta(p)X_{U(i)} + \zeta^2(p)) \\
&= \mathrm{P}(X_{U(r)}^2 - X_{U(i)}^2 - 2\zeta(p)(X_{U(r)} - X_{U(i)}) < 0) \\
&= \mathrm{P}((X_{U(r)} - X_{U(i)})(X_{U(r)} + X_{U(i)} - 2\zeta(p)) < 0) \\
&= \mathrm{P}(X_{U(r)} + X_{U(i)} - 2\zeta(p)) > 0).
\end{aligned}$$

The last equality follows from the fact that $X_{U(r)} < X_{U(i)}$ with probability 1. The PC probability can be written as

$$\pi(r,i;p) = \int_{-\infty}^{\zeta(p)}\int_{2\zeta(p)-x}^{\infty} f_{r,i}(x,y)dydx + \int_{\zeta(p)}^{\infty}\int_{x}^{\infty} f_{r,i}(x,y)dydx\,. \tag{2.4}$$

Now, the first integral in (2.4) can be simplified as follows:

$$\int_{-\infty}^{\zeta(p)}\int_{2\zeta(p)-x}^{\infty} f_{r,i}(x,y)dydx = \int_{-\infty}^{\zeta(p)}\int_{2\zeta(p)-x}^{\infty}\frac{[H(x)]^{r-1}[H(y)-H(x)]^{i-r-1}}{(r-1)!(i-r-1)!}h(x)f(y)dydx\,.$$

Making the transformation $z = H(y) - H(x),$ we have

$$\int_{-\infty}^{\zeta(p)} \int_{2\zeta(p)-x}^{\infty} f_{r,i}(x,y)dydx = \int_{-\infty}^{\zeta(p)} \int_{H(2\zeta(p)-x)-H(x)}^{\infty} \frac{[H(x)]^{r-1} f(x)}{(r-1)!} \frac{z^{i-r-1}e^{-z}}{(i-r-1)!} dzdx .$$

Using Equation (1.1), we get

$$\int_{-\infty}^{\zeta(p)} \int_{2\zeta(p)-x}^{\infty} f_{r,i}(x,y)dydx$$

$$= \sum_{j=0}^{i-r-1} \int_{-\infty}^{\zeta(p)} \frac{[H(2\zeta(p)-x)-H(x)]^{j}[1-F(2\zeta(p)-x)]}{j!} \frac{[H(x)]^{r-1} f(x)}{[1-F(x)](r-1)!} dx$$

$$= \sum_{j=0}^{i-r-1} \frac{1}{(r-1)!\,j!} \int_{-\infty}^{\zeta(p)} [H(2\zeta(p)-x)-H(x)]^{j}[1-F(2\zeta(p)-x)][H(x)]^{r-1} h(x)dx$$

$$= \sum_{j=0}^{i-r-1} \frac{1}{(r-1)!\,j!} \int_{-\infty}^{\zeta(p)} m_{j,r}(x)dx . \tag{2.5}$$

The second integral in (2.4) can be simplified as follows:

$$\int_{\zeta(p)}^{\infty} \int_{x}^{\infty} f_{r,i}(x,y)dydx = \int_{\zeta(p)}^{\infty} \int_{x}^{\infty} \frac{[H(x)]^{r-1}[H(y)-H(x)]^{i-r-1}}{(r-1)!(i-r-1)!} h(x) f(y)dydx .$$

Using the change of variables $z = H(y) - H(x)$, we get

$$\int_{\zeta(p)}^{\infty} \int_{x}^{\infty} f_{r,i}(x,y)dydx = \int_{\zeta(p)}^{\infty} \int_{0}^{\infty} \frac{[H(x)]^{r-1} f(x)}{(r-1)!} \frac{z^{i-r-1}e^{-z}}{(i-r-1)!} dzdx$$

$$= \int_{\zeta(p)}^{\infty} \frac{[H(x)]^{r-1} f(x)}{(r-1)!} dx .$$

Let $u = H(x)$, we obtain

$$\int_{\zeta(p)}^{\infty} \int_{x}^{\infty} f_{r,i}(x,y)dydx = \int_{-\log q}^{\infty} \frac{u^{r-1}e^{-u}}{(r-1)!} du .$$

Applying (1.1), we get

$$\int_{\zeta(p)}^{\infty} \int_{x}^{\infty} f_{r,i}(x,y)dydx = q \sum_{j=0}^{r-1} \frac{(-\log q)^{j}}{j!} . \tag{2.6}$$

The result in (2.1) follows by substituting the expressions in (2.5) and (2.6) into (2.4).

For $1 \le i < r$

$$\pi(r,i;p) = \mathrm{P}(|X_{U(r)} - \zeta(p)| < |X_{U(i)} - \zeta(p)|)$$

$$= \mathrm{P}([X_{U(r)} - \zeta(p)]^2 < [X_{U(i)} - \zeta(p)]^2)$$

$$= \mathrm{P}(X_{U(r)}^2 - 2\zeta(p)X_{U(r)} + \zeta^2(p) < X_{U(i)}^2 - 2\zeta(p)X_{U(i)} + \zeta^2(p))$$

$$= \mathrm{P}(X_{U(r)}^2 - X_{U(i)}^2 - 2\zeta(p)(X_{U(r)} - X_{U(i)}) < 0)$$

$$= \mathrm{P}((X_{U(r)} - X_{U(i)})(X_{U(r)} + X_{U(i)} - 2\zeta(p)) < 0)$$

$$= \mathrm{P}(X_{U(r)} + X_{U(i)} - 2\zeta(p)) < 0).$$

The last equality holds because $X_{U(r)} > X_{U(i)}$ with probability 1.

$$\pi(r,i;p) = \int_{-\infty}^{\zeta(p)} \int_{x}^{\zeta(p)} f_{i,r}(x,y)dydx + \int_{-\infty}^{\zeta(p)} \int_{\zeta(p)}^{2\zeta(p)-x} f_{i,r}(x,y)dydx. \tag{2.7}$$

Taking the first integral in (2.7) and making the transformation $w = H(y) - H(x)$, we have

$$\int_{-\infty}^{\zeta(p)} \int_{x}^{\zeta(p)} f_{i,r}(x,y)dydx = \int_{-\infty}^{\zeta(p)} \int_{0}^{H(\zeta(p))-H(x)} \frac{[H(x)]^{i-1} f(x)}{(i-1)!} \frac{w^{r-i-1}e^{-w}}{(r-i-1)!} dwdx$$

$$= \int_{-\infty}^{\zeta(p)} \frac{[H(x)]^{i-1} f(x)}{(i-1)!} \left[1 - \int_{H(\zeta(p))-H(x)}^{\infty} \frac{w^{r-i-1}e^{-w}}{(r-i-1)!} dw\right] dx.$$

Using Equation (1.1), the above integral becomes

$$\int_{-\infty}^{\zeta(p)} \int_{x}^{\zeta(p)} f_{i,r}(x,y)dydx = \int_{-\infty}^{\zeta(p)} \frac{[H(x)]^{i-1} f(x)}{(i-1)!} \left[1 - \sum_{j=0}^{r-i-1} \frac{[H(\zeta(p)) - H(x)]^j}{j!} e^{-H(\zeta(p))+H(x)}\right] dx$$

$$= \int_{-\infty}^{\zeta(p)} \frac{[H(x)]^{i-1} f(x)}{(i-1)!} dx - q \sum_{j=0}^{r-i-1} \frac{1}{j!(i-1)!} \int_{-\infty}^{\zeta(p)} [H(\zeta(p)) - H(x)]^j [H(x)]^{i-1} h(x)dx.$$

Making the change of variables *w = H(x)*, we obtain

$$\int_{-\infty}^{\zeta(p)}\int_{x}^{\zeta(p)} f_{i,r}(x,y)dydx = \int_{o}^{-\log q}\frac{w^{i-1}e^{-w}}{(i-1)!}dw - q\sum_{j=0}^{r-i-1}\frac{1}{j!(i-1)!}\int_{0}^{-\log q} w^{i-1}[H(\zeta(p))-w]^{j}\,dw$$

$$= 1-\int_{-\log q}^{\infty}\frac{w^{i-1}e^{-w}}{(i-1)!}dw - q\sum_{j=0}^{r-i-1}\frac{1}{j!(i-1)!}\int_{0}^{-\log q} w^{i-1}[H(\zeta(p))-w]^{j}\,dw$$

$$= 1-q\sum_{j=0}^{i-1}\frac{(-\log q)^{j}}{j!} - q\sum_{j=0}^{r-i-1}\frac{1}{j!(i-1)!}\int_{0}^{-\log q} w^{i-1}\left[H(\zeta(p)-w\right]^{j}\,dw. \qquad (2.8)$$

By taking the second integral in Equation (2.7) and making the transformation $w = H(y)-H(x)$, we have

$$\int_{-\infty}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-x} f_{i,r}(x,y)dydx = \int_{-\infty}^{\zeta(p)}\int_{H(\zeta(p))-H(x)}^{H(2\zeta(p)-x)-H(x)}\frac{[H(x)]^{i-1}f(x)}{(i-1)!}\frac{w^{r-i-1}e^{-w}}{(r-i-1)!}dwdx$$

$$= \int_{-\infty}^{\zeta(p)}\frac{[H(x)]^{i-1}f(x)}{(i-1)!}\left[\int_{H(\zeta(p))-H(x)}^{\infty}\frac{w^{r-i-1}e^{-w}}{(r-i-1)!}dw - \int_{H(2\zeta(p)-x)-H(x)}^{\infty}\frac{w^{r-i-1}e^{-w}}{(r-i-1)!}dw\right]dx$$

$$= \int_{-\infty}^{\zeta(p)}\frac{[H(x)]^{i-1}f(x)}{(i-1)!}\left[q\sum_{j=0}^{r-i-1}\frac{[H(\zeta(p))-H(x)]^{j}}{j!\left[1-F(x)\right]}\right.$$

$$\left. - \sum_{j=0}^{r-i-1}\frac{[H(2\zeta(p)-x)-H(x)]^{j}\left[1-F(2\zeta(p)-x)\right]}{j!\left[1-F(x)\right]}\right]dx\,.$$

Note that the third equality follows by applying (1.1). Now, using the change of variables $w = H(x)$, we get

$$\int_{-\infty}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-x} f_{i,r}(x,y)dydx = q\sum_{j=0}^{r-i-1}\frac{1}{j!(i-1)!}\int_{0}^{-\log q} w^{i-1}[H(\zeta(p))-w]^{j}\,dw -$$

$$\sum_{j=0}^{r-i-1}\frac{1}{(i-1)!\,j!}\int_{-\infty}^{\zeta(p)}[H(2\zeta(p)-x)-H(x)]^{j}[1-F(2\zeta(p)-x)][H(x)]^{i-1}h(x)dx$$

$$= q\sum_{j=0}^{r-i-1}\frac{1}{j!(i-1)!}\int_0^{-\log q} w^{i-1}[H(\zeta(p))-w]^j\,dw - \sum_{j=0}^{r-i-1}\frac{1}{(i-1)!\,j!}\int_{-\infty}^{\zeta(p)} m_{j,i}(x)dx\,.$$

(2.9)

Substituting (2.8) and (2.9) into (2.7) the result in (2.2) follows immediately.

Corollary 2.1

For $r = 1, 2, 3, \ldots,\ p \in (0,1)$ and $q = 1 - p$, we have

$$\pi(r, r+1, p) = q\sum_{j=0}^{r-1}\frac{(-\log q)^j}{j!} + \frac{1}{(r-1)!}\int_0^{-\log q} w^{r-1}\left[1 - F\left(-F^{-1}(1-e^{-w}) + 2\zeta(p)\right)\right]dw.$$

Proof: By setting $i = r+1$ in Equation (2.1), we have

$$\pi(r, r+1, p) = q\sum_{j=0}^{r-1}\frac{(-\log q)^j}{j!} + \frac{1}{(r-1)!}\int_{-\infty}^{\zeta(p)} [H(x)]^{r-1}[1 - F(2\zeta(p) - x)]h(x)dx\,.$$

By making the transformation $w = H(x)$, the result follows directly.

2.2 Distributions with Bounded Support

It is worth to mention that if $X_1, X_2, X_3, \ldots$ are *iid* random variables from a distribution with *cdf* F(x) of bounded support $[a,b]$, then the limits of the integral in Theorem (2.1) will be changed in order to keep $F(2\zeta(p)-x)$ defined on these limits. In fact, if $\zeta(p) \le \frac{a+b}{2}$, then we have

$$2\zeta(p) - a \le b \text{ and } 2\zeta(p) - b \le a\,. \tag{2.10}$$

From Equation (2.3) we have

$$2\zeta(p) - b < x < 2\zeta(p) - a\,. \tag{2.11}$$

So, from (2.10) and (2.11) we get $a < x < 2\zeta(p) - a$.

When $\zeta(p) > \dfrac{a+b}{2}$, we conclude that

$$2\zeta(p) - a > b \text{ and } 2\zeta(p) - b > a. \tag{2.12}$$

From (2.11) and (2.12) we obtain $2\zeta(p) - b < x < b$.

As a consequence of that if the distribution F is of bounded support $[a,b]$, we immediately get the following results.

Corollary 2.2

Let $\{X_i, i \geq 1\}$ be a sequence of *iid* random variables from continuous *cdf F(x)* of bounded support $[a,b]$. Then with $p \in (0,1)$ and $q = 1 - p$, we have for $1 \leq r < i$

$$\pi(r,i;p) = q\sum_{j=0}^{r-1} \frac{(-\log q)^j}{j!} + \sum_{j=0}^{i-r-1} \frac{1}{(r-1)!\, j!} M(j,r;p), \tag{2.13}$$

and for $1 \leq i < r$

$$\pi(r,i;p) = 1 - q\sum_{j=0}^{i-1} \frac{(-\log q)^j}{j!} - \sum_{j=0}^{r-i-1} \frac{1}{(i-1)!\, j!} M(j,i,p),$$

where

$$M(j,r;p) = \begin{cases} \displaystyle\int_a^{\zeta(p)} m_{j,r}(x)\, dx, & \text{if} \quad \zeta(p) \leq \dfrac{a+b}{2}, \\ \displaystyle\int_{2\zeta(p)-b}^{\zeta(p)} m_{j,r}(x)\, dx, & \text{if} \quad \zeta(p) > \dfrac{a+b}{2}, \end{cases}$$

with

$$m_{j,r}(x) = [H(2\zeta(p) - x) - H(x)]^j [1 - F(2\zeta(p) - x)][H(x)]^{r-1} h(x).$$

2.3 PC of Record Statistics for the Median

In this section, we consider the special case when $p = \frac{1}{2}$ and derive some results on PC of record statistics which are distribution free.

Theorem 2.2

Let F be symmetric about $\zeta(\frac{1}{2})$. Then the PC probabilities to the population median are distribution free and are expressed as follows:

For $1 \le r < i$

$$\pi(r,i,\tfrac{1}{2}) = \frac{1}{2}\sum_{j=0}^{r-1}\frac{(\log 2)^j}{j!} + \sum_{j=0}^{i-r-1}\sum_{k=0}^{j}\frac{(-1)^j}{(r-1)!(j-k)!k!}\int_0^{\log 2} w^{k+r-1}[\log(1-e^{-w})]^{j-k}[1-e^{-w}]dw, \tag{2.14}$$

and for $1 \le i < r$

$$\pi(r,i,\tfrac{1}{2}) = 1 - \frac{1}{2}\sum_{j=0}^{i-1}\frac{(\log 2)^j}{j!} - \sum_{j=0}^{r-i-1}\sum_{k=0}^{j}\frac{(-1)^j}{(i-1)!(j-k)!k!}\int_0^{\log 2} w^{k+i-1}[\log(1-e^{-w})]^{j-k}[1-e^{-w}]dw.$$

Proof: Without loss of generality, let us take $\zeta(\frac{1}{2}) = 0$

Then for $1 \le r < i$, we have

$$\int_{-\infty}^{\zeta(\frac{1}{2})} m_{j,r}dx = \int_{-\infty}^{0}\left[H(-x) - H(x)\right]^j\left[1 - F(-x)\right]\left[H(x)\right]^{r-1}h(x)dx$$

$$= \int_{-\infty}^{0}\left[-\log(1-F(-x)) + \log(1-F(x))\right]^j\left[1-F(-x)\right]\left[-\log(1-F(x))\right]^{r-1}h(x)dx$$

$$= \int_{-\infty}^{0}\left[-\log F(x) + \log(1-F(x))\right]^j\left[F(x)\right]\left[-\log(1-F(x))\right]^{r-1}h(x)\,dx.$$

The last equality follows by the symmetry of the distribution F. Now, by making the transformation $w = -\log(1-F(x))$, we get

$$\int_{-\infty}^{\zeta(\frac{1}{2})} m_{j,r} dx = (-1)^j \int_0^{\log 2} w^{r-1} \left[\log(1-e^{-w}) + w\right]^j \left[1-e^{-w}\right] dw.$$

Using the binomial expansion, we have

$$\int_{-\infty}^{\zeta(\frac{1}{2})} m_{j,r} dx = \sum_{k=0}^{j} \frac{j!(-1)^j}{k!(j-k)!} \int_0^{\log 2} w^{k+r-1} \left[\log(1-e^{-w})\right]^{j-k} \left[1-e^{-w}\right] dw. \tag{2.15}$$

Substituting (2.15) into Equation (2.1) the required result follows directly. The same steps can be followed for $1 \le i < r$.

2.4 Uniform (-1, 1) Distribution

Let us consider the uniform distribution with *pdf* and *cdf as*

$$f(x) = \frac{1}{2} \quad \text{and} \quad F(x) = \frac{1+x}{2} \quad \text{for } -1 < x < 1.$$

In this case, the *p*th quantile is $\zeta(p) = 2p - 1$ for $p \in (0,1)$. In the following theorem we derive expressions for PC probability associated with any two record statistics from this distribution.

Theorem 2.3

Let $\{X_i, i \ge 1\}$ be a sequence of *iid* random variables with uniform (-1, 1) distribution. Then with $p \in (0,1)$ and $q = 1 - p$, we have for $1 \le r < i$

$$\pi(r,i;p) = q\sum_{j=0}^{r-1} \frac{(-\log q)^j}{j!} + \sum_{j=0}^{i-r-1} \sum_{l=0}^{j} \frac{(-1)^j}{l!\,(r-1)!(j-l)!} M(j,r;p), \tag{2.16}$$

and for $1 \le i < r$

$$\pi(r,i;p) = 1 - q\sum_{j=0}^{i-1} \frac{(-\log q)^j}{j!} - \sum_{j=0}^{r-i-1} \sum_{l=0}^{j} \frac{(-1)^j}{l!\,(i-1)!(j-l)!} M(j,i;p),$$

where

$$M(j,r;p)=\begin{cases}\int_{0}^{-\log q} m_{j,r}(x)\,dx, & if \quad \zeta(p)\le 0,\\ \int_{-\log 2q}^{-\log q} m_{j,r}(x)\,dx, & if \quad \zeta(p)>0,\end{cases}$$

with

$$m_{j,r}(x)=w^{l+r-1}[\log(2q-e^{-w})]^{j-l}[2q-e^{-w}].$$

Proof: For $1\le r<i$, when $\zeta(p)\le 0$ $(p\le \frac{1}{2})$

$$\int_{-1}^{\zeta(p)} m_{j,r}(x)dx=\int_{-1}^{2p-1}\left[\log\left(\frac{1-x}{2}\right)-\log\left(\frac{1-2\zeta(p)+x}{2}\right)\right]^{j}\left[\frac{1-2\zeta(p)+x}{2}\right]\left[-\log\left(\frac{1-x}{2}\right)\right]^{r-1}\frac{1}{1-x}dx$$

Simplifying the above integral by making the transformation $w=-\log\left(\frac{1-x}{2}\right)$ we have

$$\int_{-1}^{\zeta(p)} m_{j,r}(x)dx=\int_{0}^{-\log q}\left[-w-\log\left(1-\zeta(p)-e^{-w}\right)\right]^{j}\left[1-\zeta(p)-e^{-w}\right]w^{r-1}dw$$

$$=\sum_{l=0}^{j}\binom{j}{l}(-1)^{j}\int_{0}^{-\log q} w^{l+r-1}\left[\log\left(1-\zeta(p)-e^{-w}\right)\right]^{j-l}\left[1-\zeta(p)-e^{-w}\right]dw$$

$$=\sum_{l=0}^{j}\binom{j}{l}(-1)^{j}\int_{0}^{-\log q} w^{l+r-1}\left[\log\left(2q-e^{-w}\right)\right]^{j-l}\left[2q-e^{-w}\right]dw. \tag{2.17}$$

By substituting (2.17) into Equation (2.13) the result follows directly.

When $\zeta(p)>0$ $(p>\frac{1}{2})$

$$\int_{2\zeta(p)-1}^{\zeta(p)} m_{j,r}(x)dx=\int_{4p-3}^{2p-1}\left[\log\left(\frac{1-x}{2}\right)-\log\left(\frac{1-2\zeta(p)+x}{2}\right)\right]^{j}\left[\frac{1-2\zeta(p)+x}{2}\right]\left[-\log\left(\frac{1-x}{2}\right)\right]^{r-1}\frac{1}{1-x}dx.$$

Using the transformation $w=-\log\left(\frac{1-x}{2}\right)$, the integral becomes

$$\int_{2\zeta(p)-1}^{\zeta(p)} m_{j,r}(x)dx = \int_{-\log 2q}^{-\log q} \left[-w-\log\left(1-\zeta(p)-e^{-w}\right)\right]^{j}\left[1-\zeta(p)-e^{-w}\right]w^{r-1}dw$$

$$=\sum_{l=0}^{j}\binom{j}{l}(-1)^{j}\int_{-\log 2q}^{-\log(1-p)} w^{l+r-1}\left[\log\left(1-\zeta(p)-e^{-w}\right)\right]^{j-l}\left[1-\zeta(p)-e^{-w}\right]dw$$

$$=\sum_{l=0}^{j}\binom{j}{l}(-1)^{j}\int_{-\log 2q}^{-\log q} w^{l+r-1}\left[\log\left(2q-e^{-w}\right)\right]^{j-l}\left[2q-e^{-w}\right]dw. \qquad (2.18)$$

The result follows immediately by substituting (2.18) into (2.13). In analogous manner, we can obtain the expression for $1 \le i < r$.

2.5 Exponential Distribution

Let us consider the standard exponential distribution with *cdf* $F(x) = 1-e^{-x}$ and *pdf* $f(x) = e^{-x}, x > 0$. Its *p*th quantile is $\zeta(p) = -\log(1-p)$, $\forall p \in (0,1)$.

In this case, Theorem (2.1) can be used to derive the following expressions for PC probability associated with any two record statistics from this distribution.

Theorem 2.4

Let $\{X_i, i \ge 1\}$ be a sequence of *iid* random variables from the standard exponential distribution. Then with $p \in (0,1)$ and $q = 1-p$, we have for $1 \le r < i$

$$\pi(r,i,p) = q\sum_{j=0}^{r-1}\frac{(-\log q)^{j}}{j!} + q^{2}\sum_{j=0}^{r-i-1}\sum_{k=0}^{j}\frac{(-1)^{k+j+r}\,2^{j}\,(\log q)^{j-k}\,(k+r-1)!}{k!(j-k)!(r-1)!}$$
$$-q\sum_{j=0}^{r-i-1}\sum_{k=0}^{j}\sum_{l=0}^{k+r-1}\frac{(-1)^{k+j+r}\,2^{j}\,(\log q)^{l+j-k}\,(k+r-1)!}{k!l!\,(j-k)!(r-1)!}, \qquad (2.19)$$

For $1 \le i < r$

$$\pi(r,i,p) = 1 - q\sum_{j=0}^{i-1}\frac{(-\log q)^j}{j!} - q^2\sum_{j=0}^{i-r-1}\sum_{k=0}^{j}\frac{(-1)^{k+j+i}\,2^j(\log q)^{j-k}(k+i-1)!}{k!(j-k)!(i-1)!}$$
$$+ q\sum_{j=0}^{i-r-1}\sum_{k=0}^{j}\sum_{l=0}^{k+i-1}\frac{(-1)^{k+j+i}\,2^j(\log q)^{l+j-k}(k+i-1)!}{k!\,l!\,(j-k)!(i-1)!}. \qquad (2.20)$$

Proof: For the standard exponential distribution we have $H(x) = x$, $h(x) = 1$, and $H(2\zeta(p) - x) = 2\zeta(p) - x$.

For $1 \le r < i$

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = \int_0^{-\log q}[2\zeta(p) - 2x]^j\, e^{x-2\zeta(p)}x^{r-1}dx$$

$$= 2^j q^2 \int_0^{-\log q}[\zeta(p) - x]^j\, e^x x^{r-1}dx\,.$$

Using the binomial expansion, we get

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = 2^j q^2 \sum_{k=0}^{j}\binom{j}{k}(\zeta(p))^{j-k}(-1)^k \int_0^{-\log q} x^{k+r-1}e^x dx\,.$$

Making the change of variables $w = -x$, the integral becomes

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = 2^j q^2 \sum_{k=0}^{j}\binom{j}{k}(-1)^{k+r+j}(\log q)^{j-k}\int_0^{\log q} w^{k+r-1}e^{-w}dw$$

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = 2^j q^2 \sum_{k=0}^{j}\binom{j}{k}(-1)^{k+r+j}(\log q)^{j-k}(k+r-1)!\left(1 - \int_{\log q}^{\infty}\frac{w^{k+r-1}e^{-w}}{(k+r-1)!}dw\right).$$

Applying Equation (1.1), we get

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = 2^j q^2 \sum_{k=0}^{j}\binom{j}{k}(-1)^{k+r+j}(\log q)^{j-k}(k+r-1)!\left(1 - \frac{1}{q}\sum_{l=0}^{k+r-1}\frac{(\log q)^l}{l!}\right)$$

$$\int_0^{\zeta(p)} m_{j,r}(x)dx = 2^j q^2 \sum_{k=0}^{j} \binom{j}{k} (-1)^{k+r+j} (\log q)^{j-k} (k+r-1)!$$

$$- 2^j q \sum_{k=0}^{j} \sum_{l=0}^{k+r-1} \frac{\binom{j}{k} (-1)^{k+r+j} (\log q)^{j+l-k} (k+r-1)!}{l!}. \tag{2.21}$$

The result in (2.19) follows by substituting (2.21) into Equation (2.1). In analogous manner, we can obtain the result in (2.20).

CHAPTER THREE

PITMAN CLOSENESS OF TWO –SEQUENCE RECORD STATISTICS

In this chapter we reviewed the general expressions of PC probabilities of record statistics from two sequences to population quantiles. Also, expressions of PC probabilities of record statistics to the population median for symmetric distributions are derived. Examples including uniform and exponential distributions are discussed. For more details see Raqab and Ahmadi (2010) and Raqab (2010).

3.1 PC of Record Statistics

Suppose that $\{X_i; i \geq 1\}$ is an infinite sequence of *iid* random variables with *cdf F(x)* and *pdf f(x),* and that $\{Y_i; i \geq 1\}$ is another independent sequence of *iid* random variables from the same distribution. Let us denote the PC probability of any two record statistics $X_{U(r)}$ and $Y_{U(s)}$ to a specific population quantile $\zeta(p)$ by

$$\pi(r, s; p) = P\left(\left|X_{U(r)} - \zeta(p)\right| < \left|Y_{U(s)} - \zeta(p)\right|\right).$$

In the following theorem, expressions for the probability $\pi(r, s; p)$ are derived.

Theorem 3.1

Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be two independent sequences of *iid* random variables from the same continuous *cdf F(x)* of bounded support [*a,b*]; $\zeta(0) = a$ and $\zeta(1) = b$. Then for $r, s \geq 1$, and $q = 1 - p$, $0 < p < 1$, we have for $\zeta(p) < \dfrac{\zeta(0) + \zeta(1)}{2}$

$$\pi(r, s; p) = 1 - \sum_{j=0}^{r-1} \frac{\left[H\left(2\zeta(p) - \zeta(0)\right)\right]^j}{j!}\left[1 - F\left(2\zeta(p) - \zeta(0)\right)\right] - \sum_{j=0}^{s-1}\binom{j+r-1}{j}\left(\frac{1}{2}\right)^{j+r}$$

$$+ q^2 \sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[-\log q]^i}{2^{j+r-i-1}\, i!} + \sum_{j=0}^{s-1}\frac{M(j, r; p)}{j!\,(r-1)!}, \qquad (3.1)$$

and for $\zeta(p) \geq \dfrac{\zeta(0)+\zeta(1)}{2}$

$$\pi(r,s;p) = 1 - \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r} + q^2 \sum_{j=0}^{s-1} \sum_{i=0}^{j+r-1} \binom{j+r-1}{j} \frac{[-\log q]^i}{2^{j+r-i-1}\, i\,!} + \sum_{j=0}^{s-1} \frac{M(j,r;p)}{j!\,(r-1)!}\,, \tag{3.2}$$

where

$$M(j,r;p) = \begin{cases} \displaystyle\int_0^p m_{j,r}(u)\,du - \int_p^{F(2\zeta(p)-\zeta(0))} m_{j,r}(u)\,du\,, & \text{if} \quad \zeta(p) < \dfrac{\zeta(0)+\zeta(1)}{2}, \\ \displaystyle\int_{F(2\zeta(p)-\zeta(1))}^{p} m_{j,r}(u)\,du - \int_p^1 m_{j,r}(u)\,du\,, & \text{if} \quad \zeta(p) \geq \dfrac{\zeta(0)+\zeta(1)}{2}, \end{cases}$$

with

$$m_{j,r}(u) = [-\log(1-u)]^{r-1}[H(2\zeta(p)-\zeta(u))]^j[1-F(2\zeta(p)-\zeta(u))]\,.$$

Proof: Let us rewrite the PC probability of any two records about $\zeta(p)$ as follows:

$$\begin{aligned}
\pi(r,s;p) &= P\left(\left|X_{U(r)}-\zeta(p)\right| < \left|Y_{U(s)}-\zeta(p)\right|\right) \\
&= P\left(\left|X_{U(r)}-\zeta(p)\right| < \left|Y_{U(s)}-\zeta(p)\right|,\, X_{U(r)} \leq Y_{U(s)}\right) \\
&+ P\left(\left|X_{U(r)}-\zeta(p)\right| < \left|Y_{U(s)}-\zeta(p)\right|,\, X_{U(r)} > Y_{U(s)}\right) \\
&= P\left((X_{U(r)}-\zeta(p))^2 < (Y_{U(s)}-\zeta(p))^2,\, X_{U(r)} \leq Y_{U(s)}\right) \\
&+ P\left((X_{U(r)}-\zeta(p))^2 < (Y_{U(s)}-\zeta(p))^2,\, X_{U(r)} > Y_{U(s)}\right) \\
&= P\left((X_{U(r)}-Y_{U(s)})(X_{U(r)}+Y_{U(s)}-2\zeta(p)) < 0,\, X_{U(r)} \leq Y_{U(s)}\right) \\
&+ P\left((X_{U(r)}-Y_{U(s)})(X_{U(r)}+Y_{U(s)}-2\zeta(p)) < 0,\, X_{U(r)} > Y_{U(s)}\right) \\
&= P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p) > 0,\, X_{U(r)} \leq Y_{U(s)}\right) \\
&+ P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p) < 0,\, X_{U(r)} > Y_{U(s)}\right).
\end{aligned} \tag{3.3}$$

The first term in (3.3) can be written as follows:

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)>0, X_{U(r)}\le Y_{U(s)}\right)$$

$$=P\left(Y_{U(s)}>2\zeta(p)-X_{U(r)},\, X_{U(r)}\le Y_{U(s)}\right)$$

$$=P\left(F(Y_{U(s)})>F(2\zeta(p)-X_{U(r)}),\, F(Y_{U(s)})\ge F(X_{U(r)})\right)$$

$$=P\left(U_s>F(2\zeta(p)-\zeta(U_r)), U_s\ge U_r\right).$$

The second equality follows by the fact that F is increasing function and the third equality follows from Lemma (1.2). Now, applying the conditioning argument and using the independence of the two sequences of X's and Y's, we immediately have

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)>0, X_{U(r)}\le Y_{U(s)}\right)$$

$$=\int_0^1 P\left(U_s>F(2\zeta(p)-\zeta(U_r)), U_s\ge U_r \middle| U_r=u\right) g_r(u)du$$

$$=\int_0^1 P\left(U_s>F(2\zeta(p)-\zeta(u)), U_s\ge u\right) g_r(u)du.$$

It is obvious that $F\left(2\zeta(p)-\zeta(u)\right)>u$ for $0<u<p$ and $F\left(2\zeta(p)-\zeta(u)\right)<u$ for $p<u<1$. Therefore, the above integral can be written as follows:

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)>0, X_{U(r)}\le Y_{U(s)}\right)$$

$$=\int_0^p P\left(U_s>F(2\zeta(p)-\zeta(u))\right) g_r(u)du+\int_p^1 P\left(U_s>u\right) g_r(u)du.$$

Now, if $\zeta(p)<\dfrac{\zeta(0)+\zeta(1)}{2}$ then we have $2\zeta(p)-\zeta(0)<\zeta(1)$ and $2\zeta(p)-\zeta(1)<\zeta(0)$.

This in turn implies $F\left(2\zeta(p)-\zeta(1)\right)=0<p<F\left(2\zeta(p)-\zeta(0)\right)<1$, so the probability becomes

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)>0, X_{U(r)}\le Y_{U(s)}\right)$$

$$=\int_0^p g_r(u)\left[1-G_s\left(F(2\zeta(p)-\zeta(u))\right)\right]du+\int_p^1 g_r(u)\left[1-G_s(u)\right]du$$

$$=\sum_{j=0}^{s-1}\frac{1}{j!(r-1)!}\int_0^p\left[-\log\left(1-F(2\zeta(p)-\zeta(u))\right]^j\left[1-F(2\zeta(p)-\zeta(u)\right]\left[-\log(1-u)\right]^{r-1}du$$

$$+\sum_{j=0}^{s-1}\int_p^1\frac{\left[-\log(1-u)\right]^{j+r-1}}{j!(r-1)!}(1-u)du.$$

Using the change of variables $w=-\log(1-u)$ and applying Equation (1.1), we get

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)>0, X_{U(r)}\le Y_{U(s)}\right)$$

$$=\sum_{j=0}^{s-1}\frac{1}{j!(r-1)!}\int_0^p\left[-\log\left(1-F(2\zeta(p)-\zeta(u)\right)\right]^j\left[1-F(2\zeta(p)-\zeta(u)\right]\left[-\log(1-u)\right]^{r-1}du$$

$$+\sum_{j=0}^{s-1}\int_{-\log q}^{\infty}\frac{w^{j+r-1}e^{-2w}}{j!(r-1)!}dw$$

$$=\sum_{j=0}^{s-1}\frac{1}{j!(r-1)!}\int_0^p\left[-\log\left(1-F(2\zeta(p)-\zeta(u)\right)\right]^j\left[1-F(2\zeta(p)-\zeta(u)\right]\left[-\log(1-u)\right]^{r-1}du$$

$$+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{(-\log q)^i}{2^{j+r-i}\,i!}. \tag{3.4}$$

The second term of Equation (3.3) can be written as

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)<0, X_{U(r)}>Y_{U(s)}\right)$$

$$=P\left(F(Y_{U(s)})<F(2\zeta(p)-X_{U(r)}), F(Y_{U(s)})<F(X_{U(r)})\right)$$

$$=P\left(U_s<F(2\zeta(p)-\zeta(U_r)), U_s<U_r\right).$$

Applying the conditioning argument and using the independence of X and Y sequences, we directly have

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)<0, X_{U(r)}>Y_{U(s)}\right)$$

$$=\int_0^1 P\left(U_s<F(2\zeta(p)-\zeta(U_r)), U_s<U_r / U_r=u\right)g_r(u)du$$

$$=\int_0^1 P\left(U_s<F(2\zeta(p)-\zeta(u)), U_s<u\right)g_r(u)du$$

$$=\int_0^p P\left(U_s<u\right)g_r(u)du \;+\int_p^{F(2\zeta(p)-\zeta(0))}P\left(U_s<F\left(2\zeta(p)-\zeta(u)\right)\right)g_r(u)du.$$

The last equality holds since $F\left(2\zeta(p)-\zeta(u)\right)>u$ when $0<u<p$ and $F\left(2\zeta(p)-\zeta(u)\right)<u$ when $p<u<1$. Now, the probability can be written as follows:

$$P\left(X_{U(r)}+Y_{U(s)}-2\zeta(p)<0, X_{U(r)}>Y_{U(s)}\right)$$

$$=\int_0^p G_s(u)g_r(u)\,du+\int_p^{F(2\zeta(p)-\zeta(0))}G_s\left(F(2\zeta(p)-\zeta(u))\right)g_r(u)du$$

$$= \int_0^{F(2\zeta(p)-\zeta(0))} \frac{[-\log(1-u)]^{r-1}}{(r-1)!} du - \sum_{j=0}^{s-1} \int_0^p \frac{[-\log(1-u)]^{j+r-1}}{j!(r-1)!} (1-u) du$$

$$- \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_p^{F(2\zeta(p)-\zeta(0))} [-\log(1-F(2\zeta(p)-\zeta(u))]^j [1-F(2\zeta(p)-\zeta(u)] [-\log(1-u)]^{r-1} du .$$

Making the transformation $w = -\log(1-u)$ and using Equation (1.1), we obtain

$$P\left(X_{U(r)} + Y_{U(s)} - 2\zeta(p) < 0, X_{U(r)} > Y_{U(s)}\right)$$

$$= 1 - \int_{-\log(1-F(2\zeta(p)-\zeta(0))}^{\infty} \frac{w^{r-1}e^{-w}}{(r-1)!} dw - \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r}$$

$$+ \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r} \int_{-\log q}^{\infty} \frac{2^{j+r} w^{j+r-1} e^{-2w}}{(j+r-1)!} dw$$

$$- \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_p^{F(2\zeta(p)-\zeta(0))} [-\log(1-F(2\zeta(p)-\zeta(u))]^j [1-F(2\zeta(p)-\zeta(u)] [-\log(1-u)]^{r-1} du$$

$$= 1 - \sum_{j=0}^{r-1} \frac{[-\log(1-F(2\zeta(p)-\zeta(0))]^j}{j!} [1-F(2\zeta(p)-\zeta(0))] - \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r}$$

$$+ q^2 \sum_{j=0}^{s-1} \sum_{i=0}^{j+r-1} \binom{j+r-1}{j} \frac{(-\log q)^i}{2^{j+r-i}\, i!}$$

$$- \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_p^{F(2\zeta(p)-\zeta(0))} [-\log(1-F(2\zeta(p)-\zeta(u))]^j [1-F(2\zeta(p)-\zeta(u)] [-\log(1-u)]^{r-1} du . \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), the result in (3.1) follows directly.

Now, if $\zeta(p) \geq \dfrac{\zeta(0)+\zeta(1)}{2}$ then $2\zeta(p)-\zeta(0) \geq \zeta(1)$ and $2\zeta(p)-\zeta(1) \geq \zeta(0)$. Also, it is easily checked that $0 < F(2\zeta(p)-\zeta(1)) < p < 1 = F(2\zeta(p)-\zeta(0))$. So, the first term in Equation (3.3) can be written as follows:

$$P\left(X_{U(r)} + Y_{U(s)} - 2\zeta(p) > 0, X_{U(r)} \leq Y_{U(s)}\right)$$

$$= \int_{F(2\zeta(p)-\zeta(1))}^{p} g_r(u)[1-G_s(F(2\zeta(p)-\zeta(u)))]du + \int_p^1 g_r(u)[1-G_s(u)]du$$

$$= \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_{F(2\zeta(p)-\zeta(1))}^{p} [-\log(1-F(2\zeta(p)-\zeta(u))]^{j} [1-F(2\zeta(p)-\zeta(u)][-\log(1-u)]^{r-1} du$$

$$+ q^2 \sum_{j=0}^{s-1} \sum_{i=0}^{j+r-1} \binom{j+r-1}{j} \frac{(-\log q)^i}{2^{j+r-i}\, i!}. \tag{3.6}$$

The second term in (3.3) can be evaluated as follows:

$$P\left(X_{U(r)} + Y_{U(s)} - 2\zeta(p) < 0, X_{U(r)} > Y_{U(s)}\right)$$

$$= \int_0^p G_s(u) g_r(u)\, du + \int_p^1 G_s(F(2\zeta(p)-\zeta(u))) g_r(u) du$$

$$= 1 - \sum_{j=0}^{s-1} \int_0^p \frac{[-\log(1-u)]^{j+r-1}(1-u)}{j!(r-1)!} du$$

$$- \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_p^1 [-\log(1-F(2\zeta(p)-\zeta(u))]^{j} [1-F(2\zeta(p)-\zeta(u)][-\log(1-u)]^{r-1} du$$

$$= 1 - \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r} + q^2 \sum_{j=0}^{s-1} \sum_{i=0}^{j+r-1} \binom{j+r-1}{j} \frac{(-\log q)^i}{2^{j+r-i}\, i!}$$

$$- \sum_{j=0}^{s-1} \frac{1}{j!(r-1)!} \int_p^1 [-\log(1-F(2\zeta(p)-\zeta(u))]^{j} [1-F(2\zeta(p)-\zeta(u)][-\log(1-u)]^{r-1} du\,. \tag{3.7}$$

The result (3.2) follows by substituting (3.6) and (3.7) into (3.3).

Corollary 3.1

Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be two independent sequences of *iid* random variables from the same *cdf F(x)* of unbounded support. Then for r, s ≥ 1 and $q = 1 - p$, $0 < p < 1$, we have

$$\pi(r,s;p) = 1 - \sum_{j=0}^{s-1} \binom{j+r-1}{j} \left(\frac{1}{2}\right)^{j+r} + q^2 \sum_{j=0}^{s-1} \sum_{i=0}^{j+r-1} \binom{j+r-1}{j} \frac{[-\log q]^i}{2^{j+r-i-1}\, i!}$$

$$+ \sum_{j=0}^{s-1} \frac{M(j,r;p)}{j!\,(r-1)!}, \tag{3.8}$$

where

$$M(j,r;p)=\int_0^p m_{j,r}(y)dy-\int_p^1 m_{j,r}(y)dy.$$

Proof: For unbounded support of *F*, we have $\zeta(0)\to-\infty$ and $\zeta(1)\to\infty$. Therefore, $F(2\zeta(p)-\zeta(0))\to 1$ and $F(2\zeta(p)-\zeta(1))\to 0$. As a result of that

$$\sum_{j=0}^{r-1}\frac{[H(2\zeta(p)-\zeta(0))]^j}{j!}[1-F(2\zeta(p)-\zeta(0))]\to 0$$

Applying the arguments used in the proof on Equation (3.1), the required result in the corollary follows directly.

Corollary 3.2

Let $\{X_i;i\geq 1\}$ and $\{Y_i;i\geq 1\}$ be two independent sequences of *iid* random variables from the same *cdf F(x)* with 0 < x < ∞. Then for r, s ≥ 1 and $q=1-p$, $0<p<1$, we have

$$\pi(r,s;p)=1-\sum_{j=0}^{s-1}\binom{j+r-1}{j}\left(\frac{1}{2}\right)^{j+r}+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[-\log q]^i}{2^{j+r-i-1}\,i!}$$

$$+\sum_{j=0}^{r-1}\frac{[H(2\zeta(p))]^j}{j!}[1-F(2\zeta(p))]+\sum_{j=0}^{s-1}\frac{M(j,r;p)}{j!\,(r-1)!}, \qquad (3.9)$$

where

$$M(j,r;p)=\int_0^p m_{j,r}(y)dy-\int_p^{F(2\zeta(p))} m_{j,r}(y)dy.$$

Proof: For 0 < x < ∞ we have $\zeta(0)\to 0$ and $\zeta(1)\to\infty$. Applying this on Equation (3.1), the required result follows directly.

3.2 PC of Record Statistics for the Median

In this section, for symmetric distribution about $\zeta(\frac{1}{2})$, we explain how the record statistics can be closest to the population median $p=\frac{1}{2}$. Here we derive some results on PC of record statistics which are distribution free.

Theorem 3.2

Let the distribution F be symmetric about $\zeta(\frac{1}{2})$. Then, the probabilities of PC to the population median are distribution-free and are expressed as

$$\pi(r,s;\tfrac{1}{2}) = 1-\sum_{j=0}^{s-1}\binom{j+r-1}{j}\left(\frac{1}{2}\right)^{j+r} + \frac{1}{2}\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[\log 2]^i}{2^{j+r-i}\, i!}$$

$$+\sum_{j=0}^{s-1}\frac{M(j,r;\frac{1}{2})}{j!\,(r-1)!},$$

where

$$M(j,r;\tfrac{1}{2}) = \int_0^{\frac{1}{2}} m_{j,r}(y)dy - \int_{\frac{1}{2}}^{1} m_{j,r}(y)dy,$$

with

$$m_{j,r}(u) = u\,[-\log(1-u)]^{r-1}[-\log u]^j.$$

Proof: Without loss of generality, let us take $\zeta(\frac{1}{2})=0$, then

$$F\big(2\zeta(\tfrac{1}{2})-\zeta(u)\big) = F(-\zeta(u)).$$

Since F is symmetric about $\zeta(\frac{1}{2})$, this turns out that

$$F\big(2\zeta(\tfrac{1}{2})-\zeta(u)\big) = 1-F(\zeta(u)) = 1-u$$

So, $M(j,r;\frac{1}{2})$ in Equation (3.8) can be evaluated as follows:

$$M(j,r;\tfrac{1}{2}) = \int_0^{\frac{1}{2}} u\,[-\log(1-u)]^{r-1}[-\log u]^j\,du - \int_{\frac{1}{2}}^{1} u\,[-\log(1-u)]^{r-1}[-\log u]^j\,du. \quad (3.10)$$

On substitute (3.10) into Equation (3.8), the required result follows immediately.

3.3 Results for Uniform (-1, 1)

Let X be uniform U(-1, 1) with cdf $F(x)=\frac{1+x}{2}$, $-1<x<1$, the pth quantile is $\zeta(p)=2p-1$, $p\in(0,1)$. The PC probabilities of record statistics from this distribution are given in the following theorem.

Theorem 3.3

Let $\{X_i; i\geq 1\}$ and $\{Y_i; i\geq 1\}$ be two independent sequences of *iid* random variables from U(-1 , 1) distribution. Then, with r, s ≥ 1 and $q=1-p$, we have for $0<p<\frac{1}{2}$,

$$\pi(r,s;p)=1-\sum_{j=0}^{s-1}\binom{j+r-1}{j}\left(\frac{1}{2}\right)^{k+r}-\sum_{j=0}^{r-1}\frac{[-\log(1-2p)]^j[1-2p]}{j!}$$
$$+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[-\log q]^i}{2^{j+r-i-1}j!}+\sum_{j=0}^{s-1}\frac{M(j,r;p)}{j!(r-1)!}, \tag{3.11}$$

and for $\frac{1}{2}\leq p<1$

$$\pi(r,s;p)=1-\sum_{j=0}^{s-1}\binom{j+r-1}{j}\left(\frac{1}{2}\right)^{k+r}+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[-\log q]^i}{2^{j+r-i-1}j!}+\sum_{j=0}^{s-1}\frac{M(j,r;p)}{j!(r-1)!}, \tag{3.12}$$

where

$$M(j,r;p)=\begin{cases}\displaystyle\int_{-\log q}^{-\log(2q-1)}m_{j,r}(y)dy-\int_{0}^{-\log q}m_{j,r}(y)dy, & if\quad 0<p<\frac{1}{2},\\ \displaystyle\int_{-\log q}^{\infty}m_{j,r}(y)dy-\int_{-\log 2q}^{-\log q}m_{j,r}(y)dy, & if\quad \frac{1}{2}\leq p<1,\end{cases}$$

with

$$m_{j,r}(y)=y^j e^{-2y}\left(-\log(2q-e^{-y})\right)^{r-1}.$$

Proof: For the uniform $(-1,1)$ distribution, we have $\zeta(u)=2u-1$, $\zeta(p)=2p-1$, $F\left(2\zeta(p)-\zeta(u)\right)=2p-u$, $F\left(2\zeta(p)-\zeta(0)\right)=2p$ and $F\left(2\zeta(p)-\zeta(1)\right)=2p-1$. From Equation (3.1), we have for $0<p<\frac{1}{2}$

$$M(j,s;p)=\int_0^p [-\log(1-2p+u)]^j [1-2p+u][-\log(1-u)]^{r-1}du$$
$$-\int_p^{2p} [-\log(1-2p+u)]^j [1-2p+u][-\log(1-u)]^{r-1}du.$$

Let $w=-\log(1-2p+u)$, we get

$$M(j,s;p)=\int_{-\log q}^{-\log 2q-1} w^j e^{-2w}\left[-\log(2q-e^{-w})\right]^{r-1}du-\int_0^{-\log q} w^j e^{-2w}\left[-\log(2q-e^{-w})\right]^{r-1}du. \tag{3.13}$$

We obtain the result in (3.11) by substituting (3.13) into (3.1).

For $\frac{1}{2}\le p<1$

$$M(j,s;p)=\int_{2p-1}^{p} [-\log(1-2p+u)]^j [1-2p+u][-\log(1-u)]^{r-1}du$$
$$-\int_p^{1} [-\log(1-2p+u)]^j [1-2p+u][-\log(1-u)]^{r-1}du.$$

Using the transformation $w=-\log(1-2p+u)$, we obtain

$$M(j,s;p)=\int_{-\log q}^{\infty} w^j e^{-2w}\left[-\log(2q-e^{-w})\right]^{r-1}du-\int_{-\log 2q}^{-\log q} w^j e^{-2w}\left[-\log(2q-e^{-w})\right]^{r-1}du. \tag{3.14}$$

The required result in (3.12) follows by substituting (3.14) into (3.2).

3.4 Results for the Exponential Distribution

Let X be the standard exponential distribution with *cdf* $F(x)=1-e^{-x}$, x > 0. In this case, the *p*th quantile is $\zeta(p)=-\log(1-p)$, $p\in(0,1)$. Theorem (3.4) provides the PC probabilities of record statistics from this distribution.

Theorem 3.4

Let $\{X_i; i\ge 1\}$ and $\{Y_i; i\ge 1\}$ be two independent sequences of *iid* random variables from the standard exponential distribution. Then for r, s ≥ 1 and $q=1-p$, we have

$$\pi(r,s;p)=1-q^2\sum_{j=0}^{r-1}\frac{(-2\log q)^j}{j!}-\sum_{j=0}^{s-1}\binom{r+j-1}{j}\left(\frac{1}{2}\right)^{r+j}+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j+r-1}\binom{j+r-1}{j}\frac{[-\log q]^i}{2^{k+s-j-1}i!}$$

$$+q^2\sum_{j=0}^{s-1}\sum_{i=0}^{j}\frac{(-\log q)^{j+r}(-1)^i}{(j-i)!\,(r-1)!\,i!\,(i+r)}[2^{j-i+1}-2^{j+r}].$$

Proof: For the standard exponential distribution, we have $2\zeta(p)-\zeta(u)=\log\frac{1-u}{q^2}$, $F\big(2\zeta(p)-\zeta(u)\big)=1-\frac{q^2}{1-u}$ and $F(2\zeta(p))=1-q^2$. Therefore,

$$M(j,r;p)=\int_0^p\left(-\log\frac{q^2}{1-u}\right)^j\left(\frac{q^2}{1-u}\right)\big(-\log(1-u)\big)^{r-1}du$$
$$-\int_p^{1-q^2}\left(-\log\frac{q^2}{1-u}\right)^j\left(\frac{q^2}{1-u}\right)\big(-\log(1-u)\big)^{r-1}du.$$

Making the change of variables $w=-\log(1-u)$, we get

$$M(j,r;p)=q^2(-1)^j\int_0^{-\log q}w^{r-1}\big[w+2\log q\big]^j dw-q^2(-1)^j\int_{-\log q}^{-2\log q}w^{r-1}\big[w+2\log q\big]^j dw$$

$$=q^2(-1)^j\sum_{i=0}^{j}\binom{j}{i}(2\log q)^{j-i}\int_o^{-\log q}w^{i+r-1}dw-q^2(-1)^j\sum_{i=0}^{j}\binom{j}{i}(2\log q)^{j-i}\int_{-\log q}^{-2\log q}w^{i+r-1}dw$$

$$=q^2(-1)^j\sum_{i=0}^{j}\binom{j}{i}(2\log q)^{j-i}\left[\frac{2(-\log q)^{r+i}-(-2\log q)^{r+i}}{r+i}\right]$$

$$=q^2\sum_{i=0}^{j}\binom{j}{i}(-1)^{r+i+j}(\log q)^{j+r}\left[\frac{2^{j-i+1}-2^{j+r}}{i+r}\right]. \qquad (3.15)$$

The result follows by substituting (3.15) into Equation (3.9).

CHAPTER FOUR

PITMAN CLOSENESS OF K-RECORD STATISTICS BASED ON ONE-SEQUENCE PROBLEM

In this chapter, expressions for PC of k-record statistics to population quantiles of a location-scale family of distributions in one-sequence problem are derived. Also, some results of PC probabilities of k-record statistics to population median are obtained. Examples including uniform and exponential distributions are discussed.

4.1 PC of k-Record Statistics

Suppose that $\{X_i; i \geq 1\}$ is an infinite sequence of *iid* random variables with *cdf F(x)* and *pdf f(x)*, and that $X_{U(r,k)}$ and $X_{U(s,k)}$ are any two k-record statistics from this sequence. Let us denote the PC probability of $X_{U(r,k)}$ and $X_{U(s,k)}$ to a specific population quantile $\zeta(p)$ by

$$\pi(r,s,k;p) = \mathrm{P}\,(|\,X_{U(r,k)} - \zeta(p)\,| < |\,X_{U(s,k)} - \zeta(p)\,|).$$

In the following theorem, we derive general expressions for the PC probability $\pi(r,s,k;p)$.

Theorem 4.1

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from a continuous distribution with *cdf F(x)* of bounded support $[a,b]$. Then, with $1 \leq r < s$, $k \geq 1$ and $q = 1 - p$, we have

$$\pi(r,s,k;p) = q^k \sum_{i=0}^{r-1} \frac{k^i(-\log q)^i}{i!} + \sum_{i=0}^{s-r-1} \frac{k^{i+r}}{i!(r-1)!} N(i,r,k;p), \tag{4.1}$$

where

$$N(i,r,k;p) = \begin{cases} \int_a^{\zeta(p)} n_{i,r,k}(x)dx, & \text{if } \zeta(p) < \frac{a+b}{2}, \\ \int_{2\zeta(p)-b}^{\zeta(p)} n_{i,r,k}(x)dx, & \text{if } \zeta(p) \geq \frac{a+b}{2}, \end{cases}$$

with

$$n_{i,r,k}(x) = [H(2\zeta(p)-x) - H(x)]^i [1 - F(2\zeta(p)-x)]^k [H(x)]^{r-1} h(x).$$

Proof: We have for $1 \leq r < s$

$$\pi(r,s,k;p) = P\left(\left|X_{U(r,k)} - \zeta(p)\right| < \left|X_{U(s,k)} - \zeta(p)\right| \right)$$

$$= P\left([X_{U(r,k)} - \zeta(p)]^2 < [X_{U(s,k)} - \zeta(p)]^2 \right)$$

$$= P\left(X_{U(r,k)}^2 - 2\zeta(p)X_{U(r,k)} + \zeta^2(p) < X_{U(s,k)}^2 - 2\zeta(p)X_{U(s,k)} + \zeta^2(p) \right)$$

$$= P\left(X_{U(r,k)}^2 - X_{U(s,k)}^2 - 2\zeta(p)(X_{U(r,k)} - X_{U(s,k)}) < 0 \right)$$

$$= P\left((X_{U(r,k)} - X_{U(s,k)})(X_{U(r,k)} + X_{U(s,k)} - 2\zeta(p)) < 0 \right)$$

$$= P\left(X_{U(r,k)} + X_{U(s,k)} - 2\zeta(p) > 0 \right).$$

When $\zeta(p) < \frac{a+b}{2}$, the probability can be computed as follows:

$$\pi(r,s,k;p) = \int_a^{\zeta(p)} \int_{2\zeta(p)-x}^{2\zeta(p)-a} f_{r,s,k}(x,y)dy\,dx + \int_{\zeta(p)}^{b} \int_x^b f_{r,s,k}(x,y)dy\,dx + \int_a^{\zeta(p)} \int_{2\zeta(p)-a}^{b} f_{r,s,k}(x,y)dy\,dx. \tag{4.2}$$

Taking the first integral in (4.2), we get

$$\int_a^{\zeta(p)}\int_{2\zeta(p)-x}^{2\zeta(p)-a} f_{r,s,k}(x,y)\,dy\,dx =$$

$$\int_a^{\zeta(p)}\int_{2\zeta(p)-x}^{2\zeta(p)-a} \frac{k^s\left[H(x)\right]^{r-1} h(x)\left[H(y)-H(x)\right]^{s-r-1}\left[1-F(y)\right]^{k-1} f(y)}{(r-1)!(s-r-1)!}\,dy\,dx\,.$$

Making the transformation $w = H(y) - H(x)$, we have

$$\int_a^{\zeta(p)}\int_{2\zeta(p)-x}^{2\zeta(p)-a} f_{r,s,k}(x,y)\,dy\,dx = \int_a^{\zeta(p)}\int_{H(2\zeta(p)-x)-H(x)}^{H(2\zeta(p)-a)-H(x)} \frac{k^r\left[H(x)\right]^{r-1}\left[1-F(x)\right]^{k-1} f(x)}{(r-1)!}\,\frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!}\,dw\,dx$$

$$= \int_a^{\zeta(p)}\int_{H(2\zeta(p)-x)-H(x)}^{\infty} \frac{k^r\left[H(x)\right]^{r-1}\left[1-F(x)\right]^{k-1} f(x)}{(r-1)!}\,\frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!}\,dw\,dx$$

$$- \int_a^{\zeta(p)}\int_{H(2\zeta(p)-a)-H(x)}^{\infty} \frac{k^r\left[H(x)\right]^{r-1}\left[1-F(x)\right]^{k-1} f(x)}{(r-1)!}\,\frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!}\,dw\,dx\,.$$

Using Equation (1.1), we get

$$\int_a^{\zeta(p)}\int_{2\zeta(p)-x}^{2\zeta(p)-a} f_{r,s,k}(x,y)\,dy\,dx = \sum_{i=0}^{s-r-1}\frac{k^{i+r}}{i!(r-1)!}\int_a^{\zeta(p)}\left[H(2\zeta(p)-x)-H(x)\right]^i\left[1-F(2\zeta(p)-x)\right]^k\left[H(x)\right]^{r-1} h(x)\,dx$$

$$- \sum_{i=0}^{s-r-1}\frac{k^{i+r}}{i!(r-1)!}\int_a^{\zeta(p)}\left[H(2\zeta(p)-a)-H(x)\right]^i\left[1-F(2\zeta(p)-a)\right]^k\left[H(x)\right]^{r-1} h(x)\,dx\,. \tag{4.3}$$

Simplifying the second integral in (4.2) by using the change of variables $w = H(y) - H(x)$ as follows:

$$\int_{\zeta(p)}^{b}\int_x^b f_{r,s,k}(x,y)\,dy\,dx = \int_{\zeta(p)}^{b}\int_x^b \frac{k^s\left[H(x)\right]^{r-1} h(x)\left[H(y)-H(x)\right]^{s-r-1}\left[1-F(y)\right]^{k-1} f(y)}{(r-1)!(s-r-1)!}\,dy\,dx$$

$$= \int_{\zeta(p)}^{b}\int_0^{\infty} \frac{k^r\left[H(x)\right]^{r-1}\left[1-F(x)\right]^{k-1} f(x)}{(r-1)!}\,\frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!}\,dw dx$$

$$= \int_{\zeta(p)}^{b} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} dx.$$

Let $w = H(x)$, then we have

$$\int_{\zeta(p)}^{b} \int_{x}^{b} f_{r,s,k}(x,y)\,dy\,dx = \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} dw.$$

Using Equation (1.1), we obtain

$$\int_{\zeta(p)}^{b} \int_{x}^{b} f_{r,s,k}(x,y)\,dy\,dx = q^k \sum_{i=0}^{r-1} \frac{k^i (-\log q)^i}{i!}. \tag{4.4}$$

Taking the third integral in (4.2) and making the change of variables $w = H(y) - H(x)$, the integral becomes

$$\int_{a}^{\zeta(p)} \int_{2\zeta(p)-a}^{b} f_{r,s,k}(x,y)\,dy\,dx = \int_{a}^{\zeta(p)} \int_{H(2\zeta(p)-a)-H(x)}^{\infty} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!} dw\,dx.$$

By using (1.1), we get

$$\int_{a}^{\zeta(p)} \int_{2\zeta(p)-a}^{b} f_{r,s,k}(x,y)\,dy\,dx = \sum_{i=0}^{s-r-1} \frac{k^{i+r}}{i!(r-1)!} \int_{a}^{\zeta(p)} [H(2\zeta(p)-a) - H(x)]^i [1-F(2\zeta(p)-a)]^k [H(x)]^{r-1} h(x)\,dx. \tag{4.5}$$

The required result in (4.1) follows by substituting (4.3), (4.4) and (4.5) into Equation (4.2).

Now, when $\zeta(p) \geq \frac{a+b}{2}$, the probability $\pi(r,s,k\,;p)$ can be evaluated as follows:

$$\pi(r,s,k\,;p) = \int_{2\zeta(p)-b}^{\zeta(p)} \int_{2\zeta(p)-x}^{b} f_{r,s,k}(x,y)\,dy\,dx + \int_{\zeta(p)}^{b} \int_{x}^{b} f_{r,s,k}(x,y)\,dy\,dx. \tag{4.6}$$

Taking the first integral in (4.6) and using the change of variables $w = H(y) - H(x)$, we obtain

$$\int_{2\zeta(p)-b}^{\zeta(p)} \int_{2\zeta(p)-x}^{b} f_{r,s,k}(x,y)\,dy\,dx = \int_{2\zeta(p)-b}^{\zeta(p)} \int_{H(2\zeta(p)-x)-H(x)}^{\infty} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!} dw\,dx.$$

Applying Equation (1.1), we get

$$\int_{2\zeta(p)-b}^{\zeta(p)} \int_{2\zeta(p)-x}^{b} f_{r,s,k}(x,y)dydx = \sum_{i=0}^{s-r-1} \frac{k^{i+r}}{i!(r-1)!} \int_{2\zeta(p)-b}^{\zeta(p)} [H(2\zeta(p)-x)-H(x)]^i [1-F(2\zeta(p)-x)]^k [H(x)]^{r-1} h(x)dx$$

(4.7)

The second integral in (4.6) can be simplified by making the change of variables $w = H(y) - H(x)$ as follows:

$$\int_{\zeta(p)}^{b} \int_{x}^{b} f_{r,s,k}(x,y)dy\,dx = \int_{\zeta(p)}^{b} \int_{0}^{\infty} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^{s-r} w^{s-r-1} e^{-kw}}{(s-r-1)!} dw\,dx$$

$$= \int_{\zeta(p)}^{b} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} dx .$$

Using the transformation $w = H(x)$ and Equation (1.1), we get

$$\int_{\zeta(p)}^{b} \int_{x}^{b} f_{r,s,k}(x,y)dy\,dx = \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} dw$$

$$= q^k \sum_{i=0}^{r-1} \frac{k^i (-\log q)^i}{i!} . \qquad (4.8)$$

The result in (4.1) follows immediately by substituting (4.7) and (4.8) into Equation (4.6).

Corollary 4.1

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from a continuous distribution with *cdf F(x)* of bounded support $[a,b]$. Then, with $1 \leq s < r$, $k \geq 1$ and $q = 1 - p$, we have

$$\pi(r,s,k;p) = 1 - q^k \sum_{i=0}^{s-1} \frac{k^i (-\log q)^i}{i!} - \sum_{i=0}^{r-s-1} \frac{k^{i+s}}{i!(s-1)!} N(i,s,k;p) ,$$

where

$$N(i,s,k\,;p)=\begin{cases}\int_{a}^{\zeta(p)} n_{i,s,k}(x)dx, & if\ \zeta(p)<\dfrac{a+b}{2},\\ \int_{2\zeta(p)-b}^{\zeta(p)} n_{i,s,k}(x)dx, & if\ \zeta(p)\ge\dfrac{a+b}{2},\end{cases}$$

with

$$n_{i,s,k}(x)=\left[H(2\zeta(p)-x)-H(x)\right]^{i}\left[1-F(2\zeta(p)-x)\right]^{k}\left[H(x)\right]^{s-1}h(x).$$

Proof: For $1 \le s < r$, we have

$$\pi(r,s,k\,;p)=P\left(\left|X_{U(r,k)}-\zeta(p)\right|<\left|X_{U(s,k)}-\zeta(p)\right|\right)$$

$$=P\left([X_{U(r,k)}-\zeta(p)]^2<[X_{U(s,k)}-\zeta(p)]^2\right)$$

$$=P\left(X^2_{U(r,k)}-2\zeta(p)X_{U(r,k)}+\zeta^2(p)<X^2_{U(s,k)}-2\zeta(p)X_{U(s,k)}+\zeta^2(p)\right)$$

$$=P\left(X^2_{U(r,k)}-X^2_{U(s,k)}-2\zeta(p)\left[X_{U(r,k)}-X_{U(s,k)}\right]<0\right)$$

$$=P\left((X_{U(r,k)}-X_{U(s,k)})(X_{U(r,k)}+X_{U(s,k)}-2\zeta(p))<0\right)$$

$$=P\left(X_{U(r,k)}+X_{U(s,k)}-2\zeta(p)<0\right)$$

$$=1-P\left(X_{U(r,k)}+X_{U(s,k)}-2\zeta(p)\ge 0\right).$$

The last equality shows that, $\pi(r,s,k\,;p)$ is the complement of the corresponding probability in Theorem (4.1). So, it can be written as follows:

$$\pi(r,s,k\,;p)=1-\pi(s,r,k\,;p). \tag{4.9}$$

$\pi(s,r,k\,;p)$ can be computed using Equation (4.1) as follows:

$$\pi(s,r,k\,;p)=q^k\sum_{i=0}^{s-1}\frac{k^i(-\log q)^i}{i!}+\sum_{i=0}^{r-s-1}\frac{k^{i+s}}{i!(s-1)!}N(i,s,k\,;p). \tag{4.10}$$

By substituting (4.10) into (4.9), we immediately obtain the result.

Corollary 4.2

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from a continuous distribution with *cdf F(x)* of unbounded support. Then, with $1 \leq r < s$, $k \geq 1$ and $q = 1 - p$, we have

$$\pi(r,s,k;p) = q^k \sum_{i=0}^{r-1} \frac{k^i(-\log q)^i}{i!} + \sum_{i=0}^{s-r-1} \frac{k^{i+r}}{i!(r-1)!} \int_{-\infty}^{\zeta(p)} n_{i,r,k}(x)dx, \qquad (4.11)$$

where

$$n_{i,r,k}(x) = [H(2\zeta(p)-x) - H(x)]^i [1 - F(2\zeta(p)-x)]^k [H(x)]^{r-1} h(x).$$

Proof: For unbounded support of F, we consider $a \to -\infty$ and $b \to \infty$. Hence $2\zeta(p) - a \to \infty$ and $2\zeta(p) - b \to -\infty$. Arguments similar to those in the proof of Theorem (4.1) lead to the required expression.

Corollary 4.3

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from a continuous distribution with *cdf F(x)* and *pdf f(x)*. Then, for r = 1, 2, 3, …, $k \geq 1$ and $q = 1 - p$, we have

$$\pi(r,r+1,k;p) = q^k \sum_{i=0}^{r-1} \frac{k^i(-\log q)^i}{i!} + \frac{k^r}{(r-1)!} \int_0^{-\log q} w^{r-1} \left[1 - F(2\zeta(p) - F^{-1}(1-e^{-w})\right]^k dw.$$

Proof: By setting *s = r + 1* in Equation (4.11), we get

$$\pi(r,r+1,k;p) = q^k \sum_{i=0}^{r-1} \frac{k^i(-\log q)^i}{i!} + \frac{k^r}{(r-1)!} \int_{-\infty}^{\zeta(p)} [1 - F(2\zeta(p)-x)]^k [H(x)]^{r-1} h(x)dx.$$

By making the transformation *w = H(x)*. The result follows directly.

4.2 PC of k-Record statistics for the Median

In this section, we derive expression for the PC of k-record statistics to the population median. The so obtained result is distribution free.

Theorem 4.2

Let F be symmetric about $\zeta(\frac{1}{2})$. Then PC probabilities to the population median are distribution free and are expressed as

$$\pi(r,s,k\,;\tfrac{1}{2}) = \sum_{i=0}^{r-1} \frac{k^i(\log 2)^i}{2^k\; i\,!} + \sum_{i=0}^{s-r-1}\sum_{j=0}^{i} \frac{k^{i+r}(-1)^j}{j!(i-j)!(r-1)!} \int_0^{\log 2} n_{i,r,k}(x)dx\,,$$

where

$$n_{i,r,k}(x) = x^{j+r-1}[-\log(1-e^{-x})]^{i-j}[1-e^{-x}]^k\,.$$

Proof: Without loss of generality, let $\zeta(\frac{1}{2}) = 0$. Using Equation (4.11), we have

$$\int_{-\infty}^{0} n_{i,r,k}(x)dx = \int_{-\infty}^{0} [H(-x)-H(x)]^i [1-F(-x)]^k [H(x)]^{r-1} h(x)dx\,.$$

By applying the binomial expansion, we get

$$\int_{-\infty}^{0} n_{i,r,k}(x)dx = \sum_{j=0}^{i}\binom{i}{j}(-1)^j \int_{-\infty}^{0} [H(x)]^{j+r-1}[-\log(1-F(-x))]^{i-j}[1-F(-x)]^k h(x)dx$$

$$= \sum_{j=0}^{i}\binom{i}{j}(-1)^j \int_{-\infty}^{0} [H(x)]^{j+r-1}[-\log F(x)]^{i-j}[F(x)]^k h(x)dx\,.$$

The second equality follows by the symmetry of the distribution F. Now, using the change of variables $w = H(x)$, the result follows directly.

4.3 PC of k-Record Statistics from Uniform (-1 , 1) Distribution

In this section we deal with the uniform distribution with *cdf* $F(x) = \dfrac{1+x}{2}$ and *pdf* $f(x) = \frac{1}{2}$, $-1 < x < 1$. In this case, the pth quantile is $\zeta(p) = 2p-1$. The following theorem

shows expressions for the PC probabilities associated with any two k-record statistics from $U(-1,1)$ distribution.

Theorem 4.3

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from uniform (-1,1) distribution. Then, with $1 \leq r < s$, $k \geq 1$ and $q = 1 - p$, we have

$$\pi(r,s,k;p) = q^k \sum_{i=0}^{r-1} \frac{k^i(-\log q)^i}{i!} + \sum_{i=0}^{s-r-1}\sum_{j=0}^{i} \frac{(-1)^i k^{i+r}}{j!(i-j)!(r-1)!} N(i,r,k;p), \tag{4.12}$$

where

$$N(i,r,k;p) = \begin{cases} \int_0^{-\log q} n_{i,r,k}(w)\,dw, & \text{if } 0 < p < \frac{1}{2}, \\ \int_{-\log 2q}^{-\log q} n_{i,r,k}(w)\,dw, & \text{if } \frac{1}{2} \leq p < 1, \end{cases}$$

with

$$n_{i,r,k}(w) = w^{j+r-1}\left[\log\left(2q - e^{-w}\right)\right]^{i-j}\left[2q - e^{-w}\right]^k.$$

Proof: If $\zeta(p) < 0$ $(p < \frac{1}{2})$, we have

$$N(i,r,k;p) = \int_{-1}^{2p-1}\left[\log\left(\frac{1-x}{2}\right) - \log\left(\frac{1-2\zeta(p)+x}{2}\right)\right]^i \left[\frac{1-2\zeta(p)+x}{2}\right]^k \left[-\log\left(\frac{1-x}{2}\right)\right]^{r-1} \frac{1}{1-x}\,dx.$$

Taking the transformation $w = -\log\dfrac{1-x}{2}$, we get

$$N(i,r,k;p) = \int_0^{-\log q} w^{r-1}\left[-w - \log\left(1 - \zeta(p) - e^{-w}\right)\right]^i \left[1 - \zeta(p) - e^{-w}\right]^k dw.$$

By using the binomial expansion, the expression becomes

$$N(i,r,k;p) = \frac{i!\,(-1)^i}{j!\,(i-j)!} \int_0^{-\log q} w^{j+r-1}\left[\log\left(1 - \zeta(p) - e^{-w}\right)\right]^{i-j}\left[1 - \zeta(p) - e^{-w}\right]^k dw.$$

By setting $1-\zeta(p)=2q$ the integral becomes

$$N(i,r,k\,;p)=\frac{i!\,(-1)^i}{j!\,(i-j)!}\int_0^{-\log q} w^{j+r-1}\left[\log\left(2q-e^{-w}\right)\right]^{i-j}\left[2q-e^{-w}\right]^k dw. \tag{4.13}$$

By substituting (4.13) into Equation (4.1), we obtain the required result. In analogous manner, the result for $\zeta(p)\geq 0\ (p\geq\frac{1}{2})$ can be obtained.

Corollary 4.4

Let $\{X_i; i\geq 1\}$ be a sequence of *iid* random variables with uniform (-1,1) distribution. Then, for r = 1, 2, 3, …, k ≥ 1 and $q=1-p$, we have

$$\pi(r,r+1,k\,;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+k^r\sum_{l=0}^{k}\frac{\binom{k}{l}(2q)^{k-l}(-1)^l}{l^r}-q^k k^r\sum_{l=0}^{k}\sum_{j=0}^{r-1}\frac{\binom{k}{l}2^{k-l}(-1)^l(-\log q)^j}{l^{r-j}\,j!},$$

$$for\ 0<p<\tfrac{1}{2}\ , \tag{4.14}$$

and

$$\pi(r,r+1,k\,;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+q^k k^r\sum_{l=0}^{k}\sum_{j=0}^{r-1}\frac{\binom{k}{l}2^k(-1)^l(-\log 2q)^j}{l^{r-j}\,j!}$$
$$-q^k k^r\sum_{l=0}^{k}\sum_{j=0}^{r-1}\frac{\binom{k}{l}2^{k-l}(-1)^l(-\log q)^j}{l^{r-j}\,j!}. \qquad for\ \tfrac{1}{2}\leq p<1$$

Proof: Setting $s = r+1$ in Equation (4.12), we have for $\zeta(p)<0\ (p<\frac{1}{2})$

$$\pi(r,r+1,k\,;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+\frac{k^r}{(r-1)!}\int_0^{-\log q} w^{r-1}\left[2q-e^{-w}\right]^k dw.$$

Using the binomial expansion, we get

$$\pi(r,r+1,k;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+\sum_{l=0}^{k}\frac{\binom{k}{l}k^r(2q)^{k-l}(-1)^l}{l^r}\int_0^{-\log q}\frac{l^r w^{r-1}e^{-lw}}{(r-1)!}dw$$

$$=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+\sum_{l=0}^{k}\frac{\binom{k}{l}k^r(2q)^{k-l}(-1)^l}{l^r}\left[1-\int_{-\log q}^{\infty}\frac{l^r w^{r-1}e^{-lw}}{(r-1)!}dw\right].$$

Applying Equation (1.1), we get

$$\pi(r,r+1,k;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+\sum_{l=0}^{k}\frac{\binom{k}{l}k^r(2q)^{k-l}(-1)^l}{l^r}\left[1-q^l\sum_{j=0}^{r-1}\frac{(-l\log q)^j}{j!}\right],$$

which is equivalent to the result in (4.14).

Now, we have for $\zeta(p)\ge 0$ $(p\ge \frac{1}{2})$

$$\pi(r,r+1,k;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}+\frac{k^r}{(r-1)!}\int_{-\log 2q}^{-\log q}w^{r-1}\left[2q-e^{-w}\right]^k dw$$

$$\pi(r,r+1,k;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}$$

$$+\sum_{l=0}^{k}\frac{\binom{k}{l}k^r(2q)^{k-l}(-1)^l}{l^r}\left[\int_{-\log 2q}^{\infty}\frac{l^r w^{r-1}e^{-lw}}{(r-1)!}dw-\int_{-\log q}^{\infty}\frac{l^r w^{r-1}e^{-lw}}{(r-1)!}dw\right].$$

Using Equation (1.1), we get

$$\pi(r,r+1,k;p)=q^k\sum_{i=0}^{r-1}\frac{q^i(-\log q)^i}{i!}$$

$$+\sum_{l=0}^{k}\frac{\binom{k}{l}k^r(2q)^{k-l}(-1)^l}{l^r}\left[(2q)^l\sum_{j=0}^{r-1}\frac{(-l\log 2q)^j}{j!}-q^l\sum_{j=0}^{r-1}\frac{(-l\log q)^j}{j!}\right].$$

This shows corollary (4.4).

Table 1 presents the values of PC probabilities for the uniform (-1,1) distribution based on one-sample problem for some selected values of r, s and p with k = 2, 3, 4 and 5,

respectively. Table 1 shows that, for r < s, $\pi(r,s,k;p)$ is increasing in r for fixed s, p and k and decreasing in p for fixed s, r and k. For s < r, $\pi(r,s,k;p)$ can be computed by using the fact that $\pi(r,s,k;p) = 1 - \pi(s,r,k;p)$.

Table 1: PC Probabilities for Uniform (-1,1) for Some Selected Values of *r*, *s*, *k* and *p*.

k	*r*	*s*	*p = 0.10*	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*
2	**1**	**2**	0.9627	0.7946	0.3863	0.0966	0.0155	0.0039
		3	0.9948	0.9239	0.5173	0.1293	0.0207	0.0052
		5	1.0000	0.9942	0.7278	0.1820	0.0291	0.0073
		8	1.0000	1.0000	0.9015	0.2254	0.0361	0.0090
		10	1.0000	1.0000	0.9530	0.2382	0.0381	0.0095
	3	**4**	0.9999	0.9946	0.9110	0.5748	0.2109	0.0839
		5	1.0000	0.9988	0.9537	0.6511	0.2525	0.1027
		8	1.0000	1.0000	0.9947	0.7763	0.3305	0.1396
		10	1.0000	1.0000	0.9989	0.8093	0.3542	0.1512
	5	**6**	1.0000	0.9999	0.9945	0.9020	0.5864	0.3433
		8	1.0000	1.0000	0.9992	0.9535	0.6835	0.4242
		10	1.0000	1.0000	0.9999	0.9734	0.7332	0.4692
3	**1**	**2**	0.9244	0.6472	0.2044	0.0256	0.0016	0.0002
		3	0.9847	0.8215	0.2947	0.0368	0.0024	0.0003
		5	0.9997	0.9727	0.4821	0.0603	0.0039	0.0005
		8	1.0000	0.9994	0.7153	0.0894	0.0057	0.0007
		10	1.0000	1.0000	0.8209	0.1026	0.0066	0.0008
	3	**4**	0.9994	0.9792	0.7630	0.2838	0.0450	0.0092
		5	0.9999	0.9933	0.8433	0.3482	0.0585	0.0122
		8	1.0000	0.9999	0.9625	0.4979	0.0940	0.0205
		10	1.0000	1.0000	0.9872	0.5599	0.1107	0.0245
	5	**6**	1.0000	0.9995	0.9681	0.6785	0.2278	0.0730
		8	1.0000	1.0000	0.9919	0.7927	0.3085	0.1059
		10	1.0000	1.0000	0.9982	0.8592	0.3683	0.1321
4	**1**	**2**	0.8785	0.5131	0.1059	0.0066	0.0002	0.0000
		3	0.9681	0.7008	0.1605	0.0100	0.0003	0.0000
		5	0.9989	0.9272	0.2937	0.0184	0.0005	0.0000
		8	1.0000	0.9967	0.5098	0.0319	0.0008	0.0001
		10	1.0000	0.9997	0.6376	0.0398	0.0010	0.0001

Table 1
Continued

k	*r*	*s*	*p = 0.10*	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*
4	**3**	**4**	0.9982	0.9500	0.5883	0.1199	0.0079	0.0008
		5	0.9997	0.9797	0.6867	0.1565	0.0109	0.0011
		8	1.0000	0.9993	0.8827	0.2642	0.0208	0.0023
		10	1.0000	1.0000	0.9455	0.3239	0.0269	0.0030
	5	**6**	1.0000	0.9979	0.9060	0.4245	0.0649	0.0108
		8	1.0000	0.9998	0.9654	0.5537	0.0997	0.0178
		10	1.0000	1.0000	0.9886	0.6518	0.1324	0.0248
5	**1**	**2**	0.8281	0.4000	0.0543	0.0017	0.0000	0.0000
		3	0.9453	0.5795	0.0853	0.0027	0.0000	0.0000
		5	0.9971	0.8577	0.1700	0.0053	0.0001	0.0000
		8	1.0000	0.9883	0.3368	0.0105	0.0001	0.0000
		10	1.0000	0.9987	0.4560	0.0142	0.0001	0.0000
	3	**4**	0.9961	0.9066	0.4251	0.0458	0.0012	0.0001
		5	0.9992	0.9552	0.5212	0.0627	0.0018	0.0001
		8	1.0000	0.9972	0.7591	0.1220	0.0039	0.0002
		10	1.0000	0.9997	0.8620	0.1623	0.0055	0.0003
	5	**6**	1.0000	0.9938	0.8075	0.2294	0.0151	0.0013
		8	1.0000	0.9992	0.9087	0.3311	0.0257	0.0023
		10	1.0000	0.9999	0.9607	0.4246	0.0374	0.0036

4.4 PC of k-Record Statistics from the Standard Exponential Distribution

Let us consider the standard exponential distribution with cdf $F(x)=1-e^{-x}$ and pdf $f(x)=e^{-x}, x>0$. Its pth quantile is $\zeta(p)=-\log(1-p)$, $p\in(0,1)$. The following theorem provides the PC probabilities of k-record statistics from exponential distribution.

Theorem 4.4

Let $\{X_i; i\geq 1\}$ be a sequence of *iid* standard exponential random variables. Then, with $1\leq r<s$, $k\geq 1$ and $q=1-p$, we have

$$\pi(r,s,k;p)=q^k\sum_{i=0}^{r-1}\frac{k^i(-\log q)^i}{i!}+q^{2k}\sum_{i=0}^{s-r-1}\sum_{j=0}^{i}\frac{k^{i-j}2^i\left(-\log q\right)^{i-j}(-1)^{2j+r}(j+r-1)!}{j!(i-j)!(r-1)!}$$

$$-q^k\sum_{i=0}^{s-r-1}\sum_{j=0}^{i}\sum_{l=0}^{j+r-1}\frac{k^{i-j+l}2^i\left(\log q\right)^{l+i-j}(-1)^{i+j+r}(j+r-1)!}{l!\,j!(i-j)!(r-1)!}. \tag{4.15}$$

Proof: For the standard exponential distribution $H(x)=x, H(2\zeta(p)-x)=2\zeta(p)-x$, $1-F(2\zeta(p)-x)=q^2e^x$ and $h(x)=1$.

By using Equation (4.11), we get

$$\int_0^{\zeta(p)} n_{i,r,k}(x)\,dx=\int_0^{-\log q}\left(2\zeta(p)-2x\right)^i\left(q^2e^x\right)^k x^{r-1}dx.$$

Applying the binomial expansion, we obtain

$$\int_0^{\zeta(p)} n_{i,r,k}(x)\,dx=q^{2k}\sum_{j=0}^{i}2^i\binom{i}{j}(-\log q)^{i-j}(-1)^j\int_0^{-\log q}x^{j+r-1}e^{kx}dx.$$

By the change of variables w = -x, we have

$$\int_0^{\zeta(p)} n_{i,r,k}(x)\,dx = q^{2k}\sum_{j=0}^{i} 2^i \binom{i}{j} \left(-\log q\right)^{i-j}(-1)^{2j+r}\int_0^{\log q} w^{j+r-1}e^{-kw}dw$$

$$= q^{2k}\sum_{j=0}^{i}\frac{2^i\; i!\left(-\log q\right)^{i-j}(-1)^{2j+r}(j+r-1)!}{k^{j+r}\,j!(i-j)!}\left[1-\int_{\log q}^{\infty}\frac{k^{j+r}w^{j+r-1}e^{-kw}}{(j+r-1)!}dw\right].$$

Using Equation (1.1), we get

$$\int_0^{\zeta(p)} n_{i,r,k}(x)\,dx = q^{2k}\sum_{j=0}^{i}\frac{2^i\; i!\left(-\log q\right)^{i-j}(-1)^{2j+r}(j+r-1)!}{k^{j+r}\,j!(i-j)!}\left[1-\frac{1}{q^k}\sum_{l=0}^{j+r-1}\frac{k^l(\log q)^l}{l!}\right]$$

$$= q^{2k}\sum_{j=0}^{i}\frac{2^i\; i!\left(-\log q\right)^{i-j}(-1)^{2j+r}(j+r-1)!}{k^{j+r}\,j!(i-j)!}$$

$$-q^k\sum_{j=0}^{i}\sum_{l=0}^{j+r-1}\frac{2^i\; i!\left(\log q\right)^{l+i-j}(-1)^{i+j+r}(j+r-1)!}{k^{j+r-l}\,l!\,j!(i-j)!}. \qquad (4.16)$$

The result follows directly by substituting (4.16) into Equation (4.11).

Corollary 4.5

Let $\{X_i; i \geq 1\}$ be a sequence of *iid* random variables from the standard exponential distribution. Then, for r = 1, 2, 3, …, k ≥ 1 and $q = 1 - p$, we have

$$\pi(r, r+1, k\,; p) = q^{2k}(-1)^r + q^k\sum_{i=0}^{r-1}\frac{k^i(\log q)^i}{i!}\left[(-1)^i - (-1)^r\right].$$

Proof: Setting $s = r + 1$ in (4.15), we have

$$\pi(r, r+1, k\,; p) = q^k\sum_{i=0}^{r-1}\frac{k^i(-\log q)^i}{i!} + q^{2k}(-1)^r - q^k\sum_{i=0}^{r-1}\frac{k^i(\log q)^i(-1)^r}{i!}$$

$$\pi(r, r+1, k\,; p) = q^{2k}(-1)^r + q^k\sum_{i=0}^{r-1}\frac{k^i(\log q)^i}{i!}\left[(-1)^i - (-1)^r\right].$$

This shows corollary (4.5).

Table 2 shows the values of PC probabilities for the exponential distribution based on one-sample problem for some selected values of r, s and p with k = 2, 3, 4 and 5, respectively. From table 2, we observe that $\pi(r,s,k\,;p)$, for r < s, is increasing in r for fixed s, p and k and decreasing in p for fixed r, s and k, we can find the values of $\pi(r,s,k\,;p)$, for s < r, by using the relation $\pi(r,s,k\,;p) = 1 - \pi(s,r,k\,;p)$.

Table 2: PC Probabilities for Exponential Distribution for Some Selected Values of *r*, *s*, *k* and *p*.

k	*r*	*s*	*p = 0.10*	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*
2	**1**	**2**	0.9639	0.8086	0.4375	0.1211	0.0199	0.0050
		3	0.9952	0.9367	0.6392	0.2166	0.0388	0.0099
		5	1.0000	0.9964	0.9068	0.4986	0.1263	0.0362
		8	1.0000	1.0000	0.9955	0.8654	0.4221	0.1718
		10	1.0000	1.0000	0.9997	0.9635	0.6604	0.3506
	3	**4**	0.9999	0.9948	0.9180	0.6015	0.2320	0.0947
		5	1.0000	0.9989	0.9638	0.7188	0.3183	0.1395
		8	1.0000	1.0000	0.9985	0.9384	0.6286	0.3563
		10	1.0000	1.0000	0.9999	0.9848	0.8063	0.5471
	5	**6**	1.0000	0.9999	0.9949	0.9093	0.6068	0.3632
		8	1.0000	1.0000	0.9995	0.9727	0.7761	0.5352
		10	1.0000	1.0000	1.0000	0.9938	0.8946	0.7035
3	**1**	**2**	0.9266	0.6658	0.2344	0.0310	0.0020	0.0002
		3	0.9857	0.8464	0.3881	0.0597	0.0040	0.0005
		5	0.9997	0.9819	0.7180	0.1834	0.0152	0.0020
		8	1.0000	0.9998	0.9612	0.5319	0.0846	0.0137
		10	1.0000	1.0000	0.9936	0.7601	0.2005	0.0419
	3	**4**	0.9994	0.9800	0.7749	0.3013	0.0497	0.0103
		5	0.9999	0.9940	0.8669	0.4018	0.0758	0.0167
		8	1.0000	0.9999	0.9847	0.7208	0.2218	0.0615
		10	1.0000	1.0000	0.9977	0.8728	0.3794	0.1279
	5	**6**	1.0000	0.9995	0.9697	0.6908	0.2395	0.0783
		8	1.0000	1.0000	0.9940	0.8427	0.3836	0.1465
		10	1.0000	1.0000	0.9992	0.9352	0.5496	0.2492
4	**1**	**2**	0.8817	0.5327	0.1211	0.0078	0.0002	0.0000
		3	0.9702	0.7349	0.2166	0.0154	0.0004	0.0000
		5	0.9991	0.9492	0.4986	0.0548	0.0016	0.0001
		8	1.0000	0.9985	0.8654	0.2355	0.0112	0.0008
		10	1.0000	0.9999	0.9635	0.4431	0.0348	0.0029

Table 2
Continued

k	*r*	*s*	*p = 0.10*	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*
4	**3**	**4**	0.9983	0.9517	0.6015	0.1279	0.0087	0.0009
		5	0.9997	0.9816	0.7188	0.1846	0.0141	0.0015
		8	1.0000	0.9995	0.9384	0.4371	0.0528	0.0070
		10	1.0000	1.0000	0.9848	0.6343	0.1119	0.0175
	5	**6**	1.0000	0.9979	0.9093	0.4357	0.0687	0.0116
		8	1.0000	0.9998	0.9727	0.6144	0.1302	0.0255
		10	1.0000	1.0000	0.9938	0.7736	0.2252	0.0520
5	**1**	**2**	0.8323	0.4183	0.0615	0.0020	0.0000	0.0000
		3	0.9486	0.6183	0.1153	0.0039	0.0000	0.0000
		5	0.9975	0.8956	0.3134	0.0148	0.0002	0.0000
		8	1.0000	0.9944	0.7107	0.0830	0.0012	0.0000
		10	1.0000	0.9995	0.8869	0.1974	0.0044	0.0002
	3	**4**	0.9962	0.9093	0.4369	0.0489	0.0013	0.0001
		5	0.9993	0.9589	0.5539	0.0746	0.0023	0.0001
		8	1.0000	0.9981	0.8480	0.2189	0.0100	0.0006
		10	1.0000	0.9998	0.9467	0.3755	0.0244	0.0018
	5	**6**	1.0000	0.9939	0.8126	0.2367	0.0160	0.0014
		8	1.0000	0.9993	0.9236	0.3800	0.0343	0.0034
		10	1.0000	1.0000	0.9757	0.5457	0.0683	0.0078

CHAPTER FIVE

PITMAN CLOSENESS OF *K*- RECORD STATISTICS BASED ON TWO-SEQUENCE PROBLEM

In this chapter general expressions are derived for PC of *k*-record statistics to population quantiles of a location-scale family of distributions in two-sequence problem. Also we derive expressions of PC probabilities of k-record statistics to population median. Explicit expressions are obtained for uniform and exponential distributions and some numerical results are presented as well.

5.1 PC of *k*-Record Statistics

Suppose that $\{X_i; i \geq 1\}$ is an infinite sequence of *iid* random variables with *cdf F(x)* and *pdf f(x),* and that $\{Y_i; i \geq 1\}$ is another independent sequence of *iid* random variables from the same distribution.

Let us denote the PC probability of any two *k*-record statistics $X_{U(r,k)}$ and $Y_{U(s,k)}$ to a specific population quantile $\zeta(p)$ by

$$\pi(r,s,k\,;p) = \mathrm{P}\ (\left|X_{U(r,k)} - \zeta(p)\right| < \left|Y_{U(s,k)} - \zeta(p)\right|)\,.$$

In the following theorem, we derive expressions for the PC probability $\pi(r,s,k\,;p)$.

Theorem 5.1

Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be two independent sequences of *iid* random variables from the same continuous distribution with *cdf F(x)* of bounded support [*a* , *b*]. Then with

r, s, k ≥ 1, and $q = 1 - p$, $0 < p < 1$, we have for $\zeta(p) < \dfrac{a+b}{2}$

$$\pi(r,s,k\,;p) = \sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} + \sum_{j=0}^{s-1}\frac{k^j[H(2\zeta(p)-a)]^j[1-F(2\zeta(p)-a)]^k}{j!}$$

$$-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i-1}\,i!} + \sum_{j=0}^{r-1}\frac{k^{j+s}M(j,s,k\,;p)}{j!(s-1)!}\ , \qquad (5.1)$$

and for $\zeta(p) \geq \dfrac{a+b}{2}$

$$\pi(r,s,k\,;p) = \sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} - q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i-1}\,i!}$$
$$+\sum_{j=0}^{r-1}\frac{k^{j+s}M(j,s,k\,;p)}{j!(s-1)!}\ , \qquad (5.2)$$

where

$$M(j,s,k\,;p) = \begin{cases} \displaystyle\int_{\zeta(p)}^{2\zeta(p)-a} m_{j,s,k}(y)dy - \int_{a}^{\zeta(p)} m_{j,s,k}(y)dy, & \text{if} \quad \zeta(p) < \dfrac{a+b}{2}, \\ \displaystyle\int_{\zeta(p)}^{b} m_{j,s,k}(y)dy - \int_{2\zeta(p)-b}^{\zeta(p)} m_{j,s,k}(y)dy, & \text{if} \quad \zeta(p) \geq \dfrac{a+b}{2}, \end{cases}$$

with

$$m_{j,s,k}(y) = [H(y)]^{s-1}[H(2\zeta(p)-y)]^j[1-F(2\zeta(p)-y)]^k[1-F(y)]^{k-1}f(y)\,.$$

Proof: For r, s and k $\geq$ 1, let us write the PC probability of any two k-record statistics $X_{U(r,k)}$ and $Y_{U(s,k)}$ as

$$\pi(r,s,k;p) = \mathrm{P}\left(\left|X_{U(r,k)} - \zeta(p)\right| < \left|Y_{U(s,k)} - \zeta(p)\right|\right)$$
$$= \mathrm{P}\left(\left|X_{U(r,k)} - \zeta(p)\right| < \left|Y_{U(s,k)} - \zeta(p)\right|, X_{U(r,k)} \leq Y_{U(s,k)}\right)$$
$$+ \mathrm{P}\left(\left|X_{U(r,k)} - \zeta(p)\right| < \left|Y_{U(s,k)} - \zeta(p)\right|, X_{U(r,k)} > Y_{U(s,k)}\right)$$

$$= \mathrm{P}((X_{U(r,k)} - \zeta(p))^2 < (Y_{U(s,k)} - \zeta(p))^2, X_{U(r,k)} \le Y_{U(s,k)})$$

$$+ \mathrm{P}((X_{U(r,k)} - \zeta(p))^2 < (Y_{U(s,k)} - \zeta(p))^2, X_{U(r,k)} > Y_{U(s,k)})$$

$$= \mathrm{P}\ (X^2_{U(r,k)} - 2\zeta(p)X_{U(r,k)} + \zeta^2(p) < Y^2_{U(s,k)} - 2\zeta(p)Y_{U(s,k)} + \zeta^2(p),\ X_{U(r,k)} \le Y_{U(s,k)})$$

$$+ \mathrm{P}\ (X^2_{U(r,k)} - 2\zeta(p)X_{U(r,k)} + \zeta^2(p) < Y^2_{U(s,k)} - 2\zeta(p)Y_{U(s,k)} + \zeta^2(p),\ X_{U(r,k)} > Y_{U(s,k)})$$

$$= \mathrm{P}\ ([X^2_{U(r,k)} - Y^2_{U(s,k)} - 2\zeta(p)(X_{U(r,k)} - Y_{U(s,k)})] < 0,\ X_{U(r,k)} \le Y_{U(s,k)})$$

$$+ \mathrm{P}\ ([X^2_{U(r,k)} - Y^2_{U(s,k)} - 2\zeta(p)(X_{U(r,k)} - Y_{U(s,k)})] < 0,\ X_{U(r,k)} > Y_{U(s,k)})$$

$$= \mathrm{P}\ ((X_{U(r,k)} - Y_{U(s,k)})(X_{U(r,k)} + Y_{U(s,k)} - 2\zeta(p)) < 0,\ X_{U(r,k)} \le Y_{U(s,k)})$$

$$+ \mathrm{P}\ ((X_{U(r,k)} - Y_{U(s,k)})(X_{U(r,k)} + Y_{U(s,k)} - 2\zeta(p)) < 0,\ X_{U(r,k)} > Y_{U(s,k)})$$

$$= \mathrm{P}\ (X_{U(r,k)} + Y_{U(s,k)} - 2\zeta(p) > 0,\ X_{U(r,k)} \le Y_{U(s,k)})$$

$$+ \mathrm{P}\ (X_{U(r,k)} + Y_{U(s,k)} - 2\zeta(p) < 0,\ X_{U(r,k)} > Y_{U(s,k)})$$

$$= \mathrm{P}\ (2\zeta(p) - Y_{U(s,k)} < X_{U(r,k)} \le Y_{U(s,k)})\ + \mathrm{P}\ (Y_{U(s,k)} < X_{U(r,k)} < 2\zeta(p) - Y_{U(s,k)})\,. \qquad (5.3)$$

Note that, if $\zeta(p) < \dfrac{a+b}{2}$ then $2\zeta(p) - a < b$ and $2\zeta(p) - b < a$. This in turn implies

$$\mathrm{P}(2\zeta(p) - Y_{U(s,k)} < X_{U(r,k)} \le Y_{U(s,k)}) =$$

$$\int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)dxdy + \int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\,dxdy + \int_{\zeta(p)}^{b} \int_{\zeta(p)}^{y} f_{r,s,k}(x,y)dxdy\,. \qquad (5.4)$$

By taking the first integral in (5.4) and using the independence of the two sequences of X's and Y's, we can simplify the integral as follows:

$$\int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\ dx\,dy = \int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,k}(x) f_{s,k}(y)\,dx\,dy$$

$$= \int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dx\, dy \, .$$

Let $w = H(x)$, then we have

$$\int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\, dx\, dy = \int_{\zeta(p)}^{2\zeta(p)-a} \int_{H(2\zeta(p)-y)}^{-\log q} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\, dy$$

$$= \int_{\zeta(p)}^{2\zeta(p)-a} \int_{H(2\zeta(p)-y)}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\, dy$$

$$- \int_{\zeta(p)}^{2\zeta(p)-a} \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\, dy \, .$$

Using Equation (1.1), we get

$$\int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\, dx\, dy =$$

$$\sum_{j=0}^{r-1} \frac{k^{j+s}}{j!(s-1)!} \int_{\zeta(p)}^{2\zeta(p)-a} [H(2\zeta(p)-y)]^j [1-F(2\zeta(p)-y)]^k [H(y)]^{s-1} [1-F(y)]^{k-1} f(y) dy$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s} [-\log q]^j}{j!} \int_{\zeta(p)}^{2\zeta(p)-a} \frac{[H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dy \, .$$

Making the transformation $w = H(y)$, we have

$$\int_{\zeta(p)}^{2\zeta(p)-a} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\, dx\, dy$$

$$= \sum_{j=0}^{r-1} \frac{k^{j+s}}{j!(s-1)!} \int_{\zeta(p)}^{2\zeta(p)-a} [H(2\zeta(p)-y)]^j [1-F(2\zeta(p)-y)]^k [H(y)]^{s-1} [1-F(y)]^{k-1} f(y) dy$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s} [-\log q]^j}{j!} \int_{-\log q}^{H(2\zeta(p)-a)} \frac{w^{s-1} e^{-kw}}{(s-1)!} dw$$

$$= \sum_{j=0}^{r-1} \frac{k^{j+s}}{j!(s-1)!} \int_{\zeta(p)}^{2\zeta(p)-a} [H(2\zeta(p)-y)]^j [1-F(2\zeta(p)-y)]^k [H(y)]^{s-1} [1-F(y)]^{k-1} f(y) dy$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{-\log q}^{\infty} \frac{w^{s-1}e^{-kw}}{(s-1)!} dw + q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{H(2\zeta(p)-a)}^{\infty} \frac{w^{s-1}e^{-kw}}{(s-1)!} dw. \tag{5.5}$$

By the independence of X and Y sequences, the second integral in (5.4) becomes

$$\int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\, dx dy$$

$$= \int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,k}(x) f_{s,k}(y)\, dx\, dy$$

$$= \int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} \frac{k^r [H(x)]^{r-1}[1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^s [H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dx\, dy.$$

Making the change of variables $w = H(x)$

$$\int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\, dx\, dy = \int_{2\zeta(p)-a}^{b} \int_{0}^{-\log q} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dw dy$$

$$= \int_{2\zeta(p)-a}^{b} \frac{k^s [H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dy$$

$$- \int_{2\zeta(p)-a}^{b} \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dw dy.$$

By applying Equation (1.1), the integral above becomes

$$\int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\, dx\, dy = \int_{2\zeta(p)-a}^{b} \frac{k^s [H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dy$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{2\zeta(p)-a}^{b} \frac{[H(y)]^{s-1}[1-F(y)]^{k-1} f(y)}{(s-1)!} dy.$$

Making the transformation $w = H(y)$ and using (1.1), we get

$$\int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy = \int_{H(2\zeta(p)-a)}^{\infty} \frac{k^s w^{s-1} e^{-kw}}{(s-1)!} dw$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{H(2\zeta(p)-a)}^{\infty} \frac{w^{s-1} e^{-kw}}{(s-1)!} dw$$

$$\int_{2\zeta(p)-a}^{b} \int_{a}^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy = \sum_{j=0}^{s-1} \frac{k^j [H(2\zeta(p)-a)]^j [1-F(2\zeta(p)-a)]^k}{j!}$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{H(2\zeta(p)-a)}^{\infty} \frac{w^{s-1} e^{-kw}}{(s-1)!} dw. \tag{5.6}$$

The third integral in (5.4) can be simplified by making the transformation $w = H(x)$ and using Equation (1.1) as follows:

$$\int_{\zeta(p)}^{b} \int_{\zeta(p)}^{y} f_{r,s,k}(x,y)\,dx\,dy = \int_{\zeta(p)}^{b} \int_{\zeta(p)}^{y} f_{r,k}(x) f_{s,k}(y)\,dx dy$$

$$= \int_{\zeta(p)}^{b} \int_{\zeta(p)}^{y} \frac{k^r [H(x)]^{r-1} [1-F(x)]^{k-1} f(x)}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dx\,dy$$

$$= \int_{\zeta(p)}^{b} \int_{-\log q}^{H(y)} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\,dy$$

$$= \int_{\zeta(p)}^{b} \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\,dy$$

$$- \int_{\zeta(p)}^{b} \int_{H(y)}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\; dy$$

$$= q^k \sum_{j=0}^{r-1} \frac{k^{j+s}[-\log q]^j}{j!} \int_{\zeta(p)}^{b} \frac{[H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dy$$

$$-\sum_{j=0}^{r-1}\int_{\zeta(p)}^{b}\frac{k^{s+j}[H(y)]^{j+s-1}[1-F(y)]^{2k-1}f(y)}{(s-1)!\,j!}dy\,.$$

Making the transformation $w=H(y)$, we get

$$\int_{\zeta(p)}^{b}\int_{\zeta(p)}^{y}f_{r,s,k}(x,y)\,dx\,dy = q^{k}\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^{j}}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw-\sum_{j=0}^{r-1}\int_{-\log q}^{\infty}\frac{k^{s+j}w^{j+s-1}e^{-2kw}}{(s-1)!\,j!}dw$$

$$= q^{k}\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^{j}}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw$$

$$-\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}\int_{-\log q}^{\infty}\frac{(2k)^{s+j}\,w^{j+s-1}e^{-2kw}}{(j+s-1)!}dw$$

$$= q^{k}\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^{j}}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw$$

$$-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i}\,i!}\,. \qquad (5.7)$$

Substituting (5.5), (5.6) and (5.7) into Equation (5.4), we immediately obtain

$$\mathrm{P}(2\zeta(p)-Y_{U(s,k)}<X_{U(r,k)}\le Y_{U(s,k)})=$$

$$\sum_{j=0}^{r-1}\frac{k^{j+s}}{j!(s-1)!}\int_{\zeta(p)}^{2\zeta(p)-a}[H(2\zeta(p)-y)]^{j}[1-F(2\zeta(p)-y)]^{k}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy$$

$$+\sum_{j=0}^{s-1}\frac{k^{j}[H(2\zeta(p)-a)]^{j}[1-F(2\zeta(p)-a)]^{k}}{j!}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i}\,i!}\,. \qquad (5.8)$$

Now, the second probability in Equation (5.3) can be obtained as follows:

$$\mathrm{P}(Y_{U(s,k)}<X_{U(r,k)}<2\zeta(p)-Y_{U(s,k)})=\int_{a}^{\zeta(p)}\int_{y}^{\zeta(p)}f_{r,s,k}(x,y)\,dx\,dy+\int_{a}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-y}f_{r,s,k}(x,y)\,dx\,dy\,.$$

$$(5.9)$$

Taking the first integral in (5.9) and using the change of variables *w = H(x)*, we obtain

$$\int_a^{\zeta(p)}\int_y^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy$$

$$= \int_a^{\zeta(p)}\int_y^{\zeta(p)} \frac{k^r[H(x)]^{r-1}[1-F(x)]^{k-1}f(x)}{(r-1)!}\frac{k^s[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dx\,dy$$

$$= \int_a^{\zeta(p)}\int_{H(y)}^{-\log q} \frac{k^r w^{r-1}e^{-kw}}{(r-1)!}\frac{k^s[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$= \int_a^{\zeta(p)}\int_{H(y)}^{\infty} \frac{k^r w^{r-1}e^{-kw}}{(r-1)!}\frac{k^s[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$- \int_a^{\zeta(p)}\int_{-\log q}^{\infty} \frac{k^r w^{r-1}e^{-kw}}{(r-1)!}\frac{k^s[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$= \sum_{j=0}^{r-1}\int_a^{\zeta(p)} \frac{k^{s+j}[H(y)]^{j+s-1}[1-F(y)]^{2k-1}f(y)}{(s-1)!\,j!}dy$$

$$-q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_a^{\zeta(p)}\frac{[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dy$$

$$= \sum_{j=0}^{r-1}\frac{(j+s-1)!}{2^{j+s}(s-1)!\,j!}\int_0^{-\log q}\frac{(2k)^{s+j}w^{j+s-1}e^{-2kw}}{(j+s-1)!}dw$$

$$-q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_a^{\zeta(p)}\frac{[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dy$$

$$= \sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} - \sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}\int_{-\log q}^{\infty}\frac{(2k)^{s+j}w^{j+s-1}e^{-2kw}}{(j+s-1)!}dw$$

$$-q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_a^{\zeta(p)}\frac{[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dy$$

$$= \sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} - q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i}\,i!}$$

$$-q^{k}\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^{j}}{j!}\int_{a}^{\zeta(p)}\frac{[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dy\,. \tag{5.10}$$

The second integral in (5.9) can be simplified as follows

$$\int_{a}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-y}f_{r,s,k}(x,y)\,dx\,dy$$

$$=\int_{a}^{\zeta(p)}\int_{-\log q}^{H(2\zeta(p)-y)}\frac{k^{r}w^{r-1}e^{-kw}}{(r-1)!}\frac{k^{s}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$=\int_{a}^{\zeta(p)}\int_{-\log q}^{\infty}\frac{k^{r}w^{r-1}e^{-kw}}{(r-1)!}\frac{k^{s}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$-\int_{a}^{\zeta(p)}\int_{H(2\zeta(p)-y)}^{\infty}\frac{k^{r}w^{r-1}e^{-kw}}{(r-1)!}\frac{k^{s}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dw\,dy$$

$$=q^{k}\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^{j}}{j!}\int_{a}^{\zeta(p)}\frac{[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}dy$$

$$-\sum_{j=0}^{r-1}\frac{k^{j+s}}{j!(s-1)!}\int_{a}^{\zeta(p)}[H(2\zeta(p)-y)]^{j}[1-F(2\zeta(p)-y)]^{k}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy\,. \tag{5.11}$$

By substituting the expressions in (5.10) and (5.11) into Equation (5.9), the probability can be written as

$$\mathrm{P}\,(Y_{U(s,k)}<X_{U(r,k)}<2\zeta(p)-Y_{U(s,k)})=$$

$$\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i}\,i!}$$

$$-\sum_{j=0}^{r-1}\frac{k^{j+s}}{j!\,(s-1)!}\int_{a}^{\zeta(p)}[H(2\zeta(p)-y)]^{j}[1-F(2\zeta(p)-y)]^{k}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy\,. \tag{5.12}$$

Now, the result in Equation (5.1) can be readily obtained by substituting (5.8) and (5.12) into Equation (5.3).

If $\zeta(p) \geq \frac{a+b}{2}$ then $2\zeta(p) - a \geq b$ and $2\zeta(p) - b \geq a$. Consequently, the first term in (5.3) can be computed as follows:

$$\mathrm{P}\ (2\zeta(p) - Y_{U(s,k)} < X_{U(r,k)} \leq Y_{U(s,k)}) = \int_{\zeta(p)}^{b} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy + \int_{\zeta(p)}^{b} \int_{\zeta(p)}^{y} f_{r,s,k}(x,y)\,dx\,dy\,. \tag{5.13}$$

Let us consider the first integral in (5.13). It can be evaluated as follows:

$$\int_{\zeta(p)}^{b} \int_{2\zeta(p)-y}^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy$$

$$= \int_{\zeta(p)}^{b} \int_{H(2\zeta(p)-y)}^{-\log q} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\,dy$$

$$= \int_{\zeta(p)}^{b} \int_{H(2\zeta(p)-y)}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\,dy$$

$$- \int_{\zeta(p)}^{b} \int_{-\log q}^{\infty} \frac{k^r w^{r-1} e^{-kw}}{(r-1)!} \frac{k^s [H(y)]^{s-1} [1-F(y)]^{k-1} f(y)}{(s-1)!} dw\,dy$$

$$= \sum_{j=0}^{r-1} \frac{k^{j+s}}{j!(s-1)!} \int_{\zeta(p)}^{b} [H(2\zeta(p)-y)]^j [1-F(2\zeta(p)-y)]^k [H(y)]^{s-1} [1-F(y)]^{k-1} f(y) dy$$

$$- q^k \sum_{j=0}^{r-1} \frac{k^{j+s} [-\log q]^j}{j!} \int_{-\log q}^{\infty} \frac{w^{s-1} e^{-kw}}{(s-1)!} dw\,. \tag{5.14}$$

The first equality follows by the change of variables $w = H(x)$. The third one follows directly by applying Equation (1.1). Now, the second integral in (5.13) can be simplified as follows:

$$\int_{\zeta(p)}^{b}\int_{\zeta(p)}^{y} f_{r,s,k}(x,y)\,dx\,dy = \int_{\zeta(p)}^{b}\int_{-\log q}^{H(y)} \frac{k^r w^{r-1}e^{-kw}}{(r-1)!}\frac{k^s[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)}{(s-1)!}\,dw\,dy$$

$$= q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}\,dw$$

$$-\sum_{j=0}^{r-1}\int_{-\log q}^{\infty}\frac{k^{s+j}w^{j+s-1}e^{-2kw}}{(s-1)!\,j!}\,dw$$

$$= q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}\,dw$$

$$-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i}\,i!}. \tag{5.15}$$

Substituting the expressions in (5.14) and (5.15) into (5.13), the first term in (5.3) becomes

$$\mathrm{P}\ (2\zeta(p)-Y_{U(s,k)} < X_{U(r,k)} \le Y_{U(s,k)})$$

$$= \sum_{j=0}^{r-1}\frac{k^{j+s}}{j!(s-1)!}\int_{\zeta(p)}^{b}[H(2\zeta(p)-y)]^j[1-F(2\zeta(p)-y)]^k[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy$$

$$-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i}\,i!}. \tag{5.16}$$

The second term in Equation (5.3) can be written as the sum of the following integrals

$$\mathrm{P}\,(Y_{U(s,k)} < X_{U(r,k)} < 2\zeta(p)-Y_{U(s,k)}) = \int_{a}^{\zeta(p)}\int_{y}^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy + \int_{2\zeta(p)-b}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-y} f_{r,s,k}(x,y)\,dx\,dy$$

$$+\int_{a}^{2\zeta(p)-b}\int_{\zeta(p)}^{b} f_{r,s,k}(x,y)\,dx\,dy\,. \tag{5.17}$$

Using and change of variables and Equation (1.1), the first integral can be simplified as follows:

$$\int_a^{\zeta(p)}\int_y^{\zeta(p)} f_{r,s,k}(x,y)\,dx\,dy = \sum_{j=0}^{r-1}\int_0^{-\log q}\frac{k^{s+j}w^{j+s-1}e^{-2kw}}{(s-1)!\,j!}dw - q^k\sum_{j=0}^{r-1}\frac{k^j[-\log q]^j}{j!}$$

$$+q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw$$

$$=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} - q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i}\,i!}$$

$$-q^k\sum_{j=0}^{r-1}\frac{k^j[-\log q]^j}{j!}+q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw. \qquad (5.18)$$

Proceeding similarly, the second and third terms in (5.17) can be simplified as

$$\int_{2\zeta(p)-b}^{\zeta(p)}\int_{\zeta(p)}^{2\zeta(p)-y} f_{r,s,k}(x,y)\,dx\,dy$$

$$= q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{H(2\zeta(p)-b)}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw - q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{-\log q}^{\infty}\frac{w^{s-1}e^{-kw}}{(s-1)!}dw$$

$$-\sum_{j=0}^{r-1}\frac{k^{j+s}}{j!(s-1)!}\int_{2\zeta(p)-b}^{\zeta(p)}[H(2\zeta(p)-y)]^j[1-F(2\zeta(p)-y)]^k[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy,$$

(5.19)

and

$$\int_a^{2\zeta(p)-b}\int_{\zeta(p)}^{b} f_{r,s,k}(x,y)\,dx\,dy = q^k\sum_{j=0}^{r-1}\frac{k^j[-\log q]^j}{j!} - q^k\sum_{j=0}^{r-1}\frac{k^{j+s}[-\log q]^j}{j!}\int_{H(2\zeta(p)-b)}^{\infty}\frac{k^s w^{s-1}e^{-kw}}{(s-1)!}dw$$

(5.20)

Substituting (5.18), (5.19) and (5.20) into (5.17), we get

P $(Y_{U(s,k)} < X_{U(r,k)} < 2\zeta(p) - Y_{U(s,k)})$

$$=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s} - q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i}\,i!}$$

$$-\sum_{j=0}^{r-1}\frac{k^{j+s}}{j!(s-1)!}\int_{2\zeta(p)-b}^{\zeta(p)}[H(2\zeta(p)-y)]^{j}[1-F(2\zeta(p)-y)]^{k}[H(y)]^{s-1}[1-F(y)]^{k-1}f(y)dy\,.$$

(5.21)

Substituting (5.16) and (5.21) into (5.3), the result in (5.2) follows immediately.

Corollary 5.1

Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be two independent sequences of *iid* random variables from the same continuous distribution with *cdf F(x)* of unbounded support. Then for r, s, k $\geq$ 1, and $q = 1 - p$, 0 < p < 1, we have

$$\pi(r,s,k\,;p)=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i-1}\,i!}$$

$$+\sum_{j=0}^{r-1}\frac{k^{j+s}M(j,s,k\,;p)}{j!\,(s-1)!}, \tag{5.22}$$

where

$$M(j,s,k\,;p)=\int_{\zeta(p)}^{\infty}m_{j,s,k}(y)dy-\int_{-\infty}^{\zeta(p)}m_{j,s,k}(y)dy\,.$$

Proof: For unbounded support of F, we consider $a \to -\infty$ and $b \to \infty$. Hence $2\zeta(p) - a \to \infty$ and $2\zeta(p) - b \to -\infty$. This implies that $F(2\zeta(p) - a) \to 1$ and then

$$\sum_{j=0}^{s-1}[H(2\zeta(p)-a)]^{j}[1-F(2\zeta(p)-a)]^{k}\to 0\,.$$

Arguments similar to those in the proof of Theorem (5.1) lead to the expression in (5.22).

Corollary 5.2

Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be two independent sequences of *iid* random variables from the same continuous distribution with *cdf F(x)* , $0 < x < \infty$. Then for r, s, k ≥ 1, and $q = 1 - p$, $0 < p < 1$, we have

$$\pi(r,s,k;p) = \sum_{j=0}^{r-1} \binom{j+s-1}{j} \left(\frac{1}{2}\right)^{j+s} - q^{2k} \sum_{j=0}^{r-1} \sum_{i=0}^{j+s-1} \binom{j+s-1}{j} \frac{k^i[-\log q]^i}{2^{j+s-i-1}\, i!}$$

$$+ \sum_{j=0}^{s-1} \frac{k^j [H(2\zeta(p))]^j [1-F(2\zeta(p))]^k}{j!} + \sum_{j=0}^{r-1} \frac{k^{j+s} M(j,s,k;p)}{j!\,(s-1)!}, \qquad (5.23)$$

where

$$M(j,s,k;p) = \int_{\zeta(p)}^{2\zeta(p)} m_{j,s,k}(y)dy - \int_0^{\zeta(p)} m_{j,s,k}(y)dy .$$

Proof: The result in (5.23) follows directly by applying (5.1) by considering $a \to 0$ and $b \to \infty$.

5.2 PC of k-Record Statistics for the Median

In this section, we consider the special case $p = \frac{1}{2}$ and derive some results on PC of k-record statistics which are distribution free.

Theorem 5.2

Let F be symmetric about $\zeta(\frac{1}{2})$. Then the probabilities of PC to the population median are distribution free and are expressed as

$$\pi(r,s,k;\tfrac{1}{2}) = \sum_{j=0}^{r-1} \binom{j+s-1}{j} \left(\frac{1}{2}\right)^{j+s} - 2\sum_{j=0}^{r-1} \sum_{i=0}^{j+s-1} \binom{j+s-1}{j} \frac{k^i[\log 2]^i}{2^{j+s-2k-i}\, i!}$$

$$+\sum_{j=0}^{r-1}\frac{k^{j+s}M(j,s,k;\tfrac{1}{2})}{j!\,(s-1)!},$$

where

$$M(j,s,k;\tfrac{1}{2})=\int_{\log 2}^{\infty}m_{j,s,k}(y)dy-\int_{0}^{\log 2}m_{j,s,k}(y)dy\,,$$

with

$$m_{j,s,k}(y)=y^{s-1}e^{-ky}[-\log(1-e^{-y})]^{j}[1-e^{-y}]^{k}\,.$$

Proof: Without loss of generality, Let us take $\zeta(\tfrac{1}{2})=0$. Then, we have

$$\int_{\zeta(p)}^{\infty}m_{j,s,k}(y)dy-\int_{-\infty}^{\zeta(p)}m_{j,s,k}(y)dy$$

$$=\int_{0}^{\infty}[-\log(1-F(y))]^{s-1}[-\log(1-F(-y))]^{j}[1-F(-y)]^{k}[1-F(y)]^{k-1}f(y)dy$$

$$-\int_{-\infty}^{0}[-\log(1-F(y))]^{s-1}[-\log(1-F(-y))]^{j}[1-F(-y)]^{k}[1-F(y)]^{k-1}f(y)dy$$

$$=\int_{0}^{\infty}[-\log(1-F(y))]^{s-1}[-\log F(y)]^{j}[F(y)]^{k}[1-F(y)]^{k-1}f(y)dy$$

$$-\int_{-\infty}^{0}[-\log(1-F(y))]^{s-1}[-\log F(y)]^{j}[F(y)]^{k}[1-F(y)]^{k-1}f(y)dy\,.$$

The second equality follows by the symmetry of the distribution. Making the transformation $w=-\log(1-F(y))$, we get

$$\int_{\zeta(p)}^{\infty}m_{j,s,k}(y)dy-\int_{-\infty}^{\zeta(p)}m_{j,s,k}(y)dy=\int_{\log 2}^{\infty}w^{s-1}e^{-kw}[-\log(1-e^{-w})]^{j}[1-e^{-w}]^{k}dw$$

$$-\int_{0}^{\log 2} w^{s-1}e^{-kw}\left[-\log(1-e^{-w})\right]^{j}\left[1-e^{-w}\right]^{k} dw. \qquad (5.24)$$

Substituting (5.24) into (5.22), the expression is readily obtained.

5.3 PC of k-Record Statistics from Uniform (-1 , 1) Distribution

Let us consider the uniform distribution with $cdf\, F(x)=\dfrac{1+x}{2}$ and $pdf\ f(x)=\frac{1}{2}$, for *-1< x < 1*. In this case, the *p*th quantile is $\zeta(p)=2p-1$. The PC probabilities associated with any two k-record statistics from the 2-samples based on $U(-1,1)$ distribution are given in the following theorem.

Theorem 5.3

Let $\{X_i; i\ge 1\}$ and $\{Y_i; i\ge 1\}$ be two independent sequences of *iid* random variables from $U(-1,1)$ distribution. Then, with r, s, k ≥ 1 and $q=1-p$, we have for $0<p<\frac{1}{2}$,

$$\pi(r,s,k;p)=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i-1}\,i!}$$

$$+(2q-1)^{k}\sum_{j=0}^{s-1}\frac{k^{j}[-\log(2q-1)]^{j}}{j!}+\sum_{j=0}^{r-1}\sum_{i=0}^{k}\frac{\binom{k}{i}k^{j+s}(2q)^{k-i}(-1)^{i}}{j!\,(s-1)!}M(j,s,k;p), \qquad (5.25)$$

and for $\frac{1}{2}\le p<1$,

$$\pi(r,s,k;p)=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^{i}[-\log q]^{i}}{2^{j+s-i-1}\,i!}$$

$$+\sum_{j=0}^{r-1}\sum_{i=0}^{k}\frac{\binom{k}{i}k^{j+s}(2q)^{k-i}(-1)^{i}}{j!\,(s-1)!}M(j,s,k\,;p)\,, \tag{5.26}$$

where

$$M(j,s,k\,;p)=\begin{cases}\displaystyle\int_{-\log q}^{-\log 2q-1} m_{j,s,k}(w)dw-\int_{0}^{-\log q} m_{j,s,k}(w)dw, & if\quad 0<p<\tfrac{1}{2},\\ \displaystyle\int_{-\log q}^{\infty} m_{j,s,k}(w)dw-\int_{-\log 2q}^{-\log q} m_{j,s,k}(w)dw, & if\quad \tfrac{1}{2}\le p<1,\end{cases}$$

with

$$m_{j,s,k}(w)=w^{s-1}e^{-(i+k)w}\left[-\log(2q-e^{-w})\right]^{j}.$$

Proof: It is readily obtained that

$H(y)=-\log\left(\frac{1-y}{2}\right)$, $2\zeta(p)-a=3-4q$, $2\zeta(p)-b=1-4q$, $1-F(2\zeta(p)-a)=2q-1$ and

$$H(2\zeta(p)-a)=-\log(2q-1)\,. \tag{5.27}$$

For $\zeta(p)<0\;\;(p<\frac{1}{2})$, we have

$$\int_{1-2q}^{3-4q} m_{j,s,k}(y)dy-\int_{-1}^{1-2q} m_{j,s,k}(y)dy$$

$$=\frac{1}{2}\int_{1-2q}^{3-4q}\left[-\log\frac{1-2\zeta(p)+y}{2}\right]^{j}\left[\frac{1-2\zeta(p)+y}{2}\right]^{k}\left[-\log\frac{1-y}{2}\right]^{s-1}\left[\frac{1-y}{2}\right]^{k-1}dy$$

$$-\frac{1}{2}\int_{-1}^{1-2q}\left[-\log\frac{1-2\zeta(p)+y}{2}\right]^{j}\left[\frac{1-2\zeta(p)+y}{2}\right]^{k}\left[-\log\frac{1-y}{2}\right]^{s-1}\left[\frac{1-y}{2}\right]^{k-1}dy\,.$$

Taking the transformation $w=-\log\dfrac{1-y}{2}$, we get

$$\int_{1-2q}^{3-4q} m_{j,s,k}(y)dy-\int_{-1}^{1-2q} m_{j,s,k}(y)dy=\int_{-\log q}^{-\log 2q-1} w^{s-1}e^{-kw}\left[-\log(2q-e^{-w})\right]^{j}\left[2q-e^{-w}\right]^{k}dw$$

$$-\int_{0}^{-\log q} w^{s-1}e^{-kw}\left[-\log(2q-e^{-w})\right]^{j}\left[2q-e^{-w}\right]^{k}dw.$$

Using binomial expansion

$$\int_{1-2q}^{3-4q} m_{j,s,k}(y)dy-\int_{-1}^{1-2q} m_{j,s,k}(y)dy=\sum_{i=0}^{k}\binom{k}{i}(2q)^{k-i}(-1)^{i}\int_{-\log q}^{-\log 2q-1} w^{s-1}e^{-(i+k)w}\left[-\log(2q-e^{-w})\right]^{j}dw$$

$$-\sum_{i=0}^{k}\binom{k}{i}(2q)^{k-i}(-1)^{i}\int_{0}^{-\log q} w^{s-1}e^{-(i+k)w}\left[-\log(2q-e^{-w})\right]^{j}dw. \quad (5.28)$$

Substituting (5.27) and (5.28) into Equation (5.1), we immediately obtain the expression in (5.25). Proceeding similarly, we obtain the expression in (5.26).

Table 3 provides values of PC probabilities for the uniform $(-1,1)$ distribution in two-sequence problem for various choices of r, s and k and some selected values of p. From Table 3, it is observed that $\pi(r,s,k;p)$, for r < s, is decreasing in p for fixed r, s and k. One can compute the values of $\pi(r,s,k;p)$, for s < r, by using the relation

$$\pi(r,s,k;p) = 1 - \pi(s,r,k;p)$$

Table3: PC Probabilities for Uniform (-1,1) for Some Selected Values of *r*, *s*, *k* and *p*.

k	*r*	*S*	*p = 0.10*	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*
2	**1**	**2**	0.7451	0.6836	0.3977	0.2592	0.2502	0.2500
		3	0.8742	0.8458	0.4897	0.1606	0.1263	0.1251
		5	0.9687	0.9669	0.7408	0.1503	0.0396	0.0322
		8	0.9961	0.9961	0.9220	0.2119	0.0290	0.0083
		10	0.9990	0.9990	0.9653	0.2327	0.0337	0.0078
	3	**4**	0.6563	0.6562	0.6427	0.4838	0.3627	0.3467
		5	0.7734	0.7734	0.7641	0.5399	0.2842	0.2374
		8	0.9453	0.9453	0.9445	0.7407	0.2818	0.1216
		10	0.9807	0.9807	0.9806	0.7992	0.3266	0.1282
	5	**6**	0.6230	0.6230	0.6230	0.6012	0.4793	0.4118
		8	0.8062	0.8062	0.8061	0.7841	0.5569	0.3575
		10	0.9102	0.9102	0.9102	0.8930	0.6601	0.3990
3	**1**	**2**	0.7358	0.6044	0.3024	0.2508	0.2500	0.2500
		3	0.8717	0.7830	0.2893	0.1293	0.1250	0.1250
		5	0.9687	0.9561	0.4747	0.0549	0.0316	0.0313
		8	0.9961	0.9959	0.7425	0.0710	0.0058	0.0040
		10	0.9990	0.9990	0.8473	0.0903	0.0046	0.0012
	3	**4**	0.6562	0.6554	0.5950	0.3856	0.3448	0.3438
		5	0.7734	0.7731	0.7147	0.3439	0.2308	0.2268
		8	0.9453	0.9453	0.9321	0.4543	0.0900	0.0579
		10	0.9807	0.9807	0.9771	0.5426	0.0865	0.0275
	5	**6**	0.6230	0.6230	0.6213	0.5292	0.3954	0.3788
		8	0.8062	0.8062	0.8056	0.6763	0.2961	0.2090
		10	0.9102	0.9102	0.9101	0.8053	0.3107	0.1348
4	**1**	**2**	0.7212	0.5211	0.2670	0.2501	0.2500	0.2500
		3	0.8661	0.6916	0.1898	0.1254	0.1250	0.1250
		5	0.9685	0.9266	0.2667	0.0349	0.0313	0.0313
		8	0.9961	0.9946	0.5184	0.0206	0.0040	0.0039
		10	0.9990	0.9989	0.6598	0.0285	0.0012	0.0010

Table 3
Continued

k	r	s	P = 0.10	0.25	0.50	0.75	0.90	0.95
4	3	4	0.6562	0.6526	0.5228	0.3526	0.3438	0.3438
		5	0.7734	0.7716	0.6147	0.2568	0.2268	0.2266
		8	0.9453	0.9453	0.8804	0.2202	0.0578	0.0548
		10	0.9807	0.9807	0.9551	0.2849	0.0276	0.0196
	5	6	0.6230	0.6230	0.6125	0.4480	0.3787	0.3770
		8	0.8062	0.8062	0.8002	0.4877	0.2085	0.1945
		10	0.9102	0.9102	0.9086	0.6069	0.1348	0.0926
5	1	2	0.7018	0.4473	0.2552	0.2500	0.2500	0.2500
		3	0.8566	0.5878	0.1485	0.1250	0.1250	0.1250
		5	0.9679	0.8725	0.1425	0.0317	0.0313	0.0313
		8	0.9961	0.9900	0.3216	0.0073	0.0039	0.0039
		10	0.9990	0.9984	0.4595	0.0079	0.0010	0.0010
	3	4	0.6562	0.6458	0.4536	0.3452	0.3438	0.3438
		5	0.7734	0.7669	0.4948	0.2326	0.2266	0.2266
		8	0.9453	0.9450	0.7705	0.1062	0.0549	0.0547
		10	0.9807	0.9807	0.8913	0.1225	0.0200	0.0193
	5	6	0.6230	0.6230	0.5897	0.4010	0.3771	0.3770
		8	0.8062	0.8061	0.7790	0.3255	0.1951	0.1939
		10	0.9102	0.9102	0.8995	0.3766	0.0953	0.0899

5.4 PC of k-Record Statistics from the Standard Exponential Distribution

Let us consider the standard exponential distribution with *pdf* and *cdf* as $f(x)=e^{-x}$ and $F(x)=1-e^{-x}$ for x > 0. Its *p*th quantile is $\zeta(p)=-\log(1-p)$, $p\in(0,1)$. In this case, Theorem 4.1 can be used to derive PC probability of k-record statistics from 2-sequences from the standard exponential distribution.

Theorem 5.4

Let $\{X_i; i\geq 1\}$ and $\{Y_i; i\geq 1\}$ be two independent sequences of *iid* random variables from the standard exponential distribution. Then for r, s, k ≥ 1, $q=1-p$ and $p\in(0,1)$, we have

$$\pi(r,s,k;p)=\sum_{j=0}^{r-1}\binom{j+s-1}{j}\left(\frac{1}{2}\right)^{j+s}-q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j+s-1}\binom{j+s-1}{j}\frac{k^i[-\log q]^i}{2^{j+s-i-1}\,i!}$$

$$+q^{2k}\sum_{j=0}^{s-1}\frac{k^j(-2\log q)^j}{j!}+q^{2k}\sum_{j=0}^{r-1}\sum_{i=0}^{j}\frac{\binom{j}{i}(-\log q)^{j+s}(-1)^i}{(s-1)!\,j!\,(i+s)}\left(2^{j+s}-2^{j-i+1}\right). \tag{5.29}$$

Proof: It is obvious that $H(x)=-\log(1-(1-e^{-x}))=x$. Therefore,

$$H(\zeta(p))=\zeta(p)=-\log q \text{ and } H(2\zeta(p))=2\zeta(p)=-2\log q. \tag{5.30}$$

Now, we have

$$\int_{\zeta(p)}^{2\zeta(p)} m_{j,s,k}(y)dy-\int_{0}^{\zeta(p)} m_{j,s,k}(y)dy$$

$$=\int_{-\log q}^{-2\log q} y^{s-1}(2\zeta(p)-y)^j e^{-2k\zeta(p)}dy \;-\; \int_{0}^{-\log q} y^{s-1}(2\zeta(p)-y)^j e^{-2k\zeta(p)}dy$$

$$= q^{2k} \int_{-\log q}^{-2\log q} y^{s-1}(2\zeta(p)-y)^j\, dy \;-\; q^{2k} \int_{0}^{-\log q} y^{s-1}(2\zeta(p)-y)^j\, dy$$

$$= q^{2k} \sum_{i=0}^{j} \binom{j}{i} (2\zeta(p))^{j-i}(-1)^i \left[\int_{-\log q}^{-2\log q} y^{i+s-1} dy - \int_{0}^{-\log q} y^{i+s-1} dy \right]$$

$$= q^{2k} \sum_{i=0}^{j} \binom{j}{i} (2\zeta(p))^{j-i}(-1)^i \left[\frac{(-2\log q)^{i+s} - 2(-\log q)^{i+s}}{i+s} \right]$$

$$= q^{2k} \sum_{i=0}^{j} \frac{\binom{j}{i} (-\log q)^{j+s}(-1)^i}{i+s} \left[2^{j+s} - 2^{j-i+1} \right]. \tag{5.31}$$

Substituting (5.30) and (5.31) into (5.23), the result is readily obtained.

Table 4 presents the values of PC probabilities for the exponential distribution in two-sequence problem for various choices of r, s and k and some selected values of p. Table 4 shows that, for r < s, $\pi(r,s,k\,;p)$ is decreasing in p for fixed r, s and k. For s < r, the values of $\pi(r,s,k\,;p)$ can be computed via the relation

$$\pi(r,s,k\,;p) = 1 - \pi(s,r,k\,;p).$$

Table 4: PC Probabilities for Exponential Distribution for Some Selected Values of *r*, *s*, *k* and *p*.

k	*r*	*s*	*p* = 0.10	0.25	0.50	0.75	0.90	0.95
2	**1**	**2**	0.7454	0.6950	0.4880	0.2928	0.2526	0.2503
		3	0.8743	0.8528	0.6485	0.2725	0.1387	0.1267
		5	0.9687	0.9676	0.9099	0.5199	0.1283	0.0505
		8	0.9961	0.9961	0.9938	0.8874	0.4288	0.1588
		10	0.9990	0.9990	0.9989	0.9725	0.6804	0.3493
	3	**4**	0.6563	0.6562	0.6561	0.6490	0.5992	0.5336
		5	0.7734	0.7734	0.7734	0.7692	0.7212	0.6341
		8	0.9453	0.9453	0.9453	0.9452	0.9383	0.9077
		10	0.9807	0.9807	0.9807	0.9807	0.9798	0.9723
	5	**6**	0.6230	0.6230	0.6230	0.6165	0.5445	0.4628
		8	0.8062	0.8062	0.8061	0.8032	0.7204	0.5477
		10	0.9102	0.9102	0.9102	0.9097	0.8703	0.7244
3	**1**	**2**	0.7368	0.6252	0.3579	0.2554	0.2501	0.2500
		3	0.8720	0.8022	0.4273	0.1506	0.1254	0.1250
		5	0.9687	0.9607	0.7436	0.1848	0.0371	0.0316
		8	0.9961	0.9960	0.9698	0.5479	0.0713	0.0110
		10	0.9990	0.9990	0.9952	0.7826	0.1888	0.0317
	3	**4**	0.6562	0.6556	0.6163	0.4300	0.3487	0.3441
		5	0.7734	0.7732	0.7400	0.4505	0.2457	0.2282
		8	0.9453	0.9453	0.9419	0.7522	0.2223	0.0821
		10	0.9807	0.9807	0.9803	0.9035	0.3816	0.1083
	5	**6**	0.6230	0.6230	0.6221	0.5697	0.4233	0.3845
		8	0.8062	0.8062	0.8059	0.7570	0.4246	0.2506
		10	0.9102	0.9102	0.9102	0.8911	0.5738	0.2619
4	**1**	**2**	0.7231	0.5478	0.2928	0.2506	0.2500	0.2500
		3	0.8669	0.7243	0.2725	0.1284	0.1250	0.1250
		5	0.9685	0.9406	0.5199	0.0646	0.0315	0.0313
		8	0.9961	0.9955	0.8874	0.2262	0.0094	0.0041
		10	0.9990	0.9990	0.9725	0.4492	0.0255	0.0021

Table 4
Continued

k	r	s	p = 0.10	0.25	0.50	0.75	0.90	0.95
4	3	4	0.6562	0.6533	0.5559	0.3672	0.3440	0.3438
		5	0.7734	0.7720	0.6662	0.3017	0.2278	0.2266
		8	0.9453	0.9453	0.9217	0.4547	0.0766	0.0557
		10	0.9807	0.9807	0.9761	0.6719	0.0936	0.0246
	5	6	0.6230	0.6230	0.6165	0.4875	0.3830	0.3772
		8	0.8062	0.8062	0.8032	0.6095	0.2411	0.1972
		10	0.9102	0.9102	0.9097	0.7850	0.2389	0.1063
5	1	2	0.7048	0.4757	0.2656	0.2501	0.2500	0.2500
		3	0.8582	0.6311	0.1891	0.1254	0.1250	0.1250
		5	0.9680	0.9012	0.3214	0.0369	0.0313	0.0313
		8	0.9961	0.9932	0.7361	0.0697	0.0042	0.0039
		10	0.9990	0.9988	0.9055	0.1856	0.0029	0.0010
	3	4	0.6562	0.6476	0.4875	0.3486	0.3438	0.3438
		5	0.7734	0.7683	0.5596	0.2452	0.2266	0.2266
		8	0.9453	0.9451	0.8618	0.2194	0.0564	0.0547
		10	0.9807	0.9807	0.9564	0.3770	0.0279	0.0195
	5	6	0.6230	0.6230	0.6000	0.4225	0.3774	0.3770
		8	0.8062	0.8061	0.7907	0.4217	0.1994	0.1940
		10	0.9102	0.9102	0.9059	0.5697	0.1150	0.0906

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مقارنات تقارب بيتمان للبيانات الرتبية

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ملخص

في دراسات سابقة تم حساب احتمال تقارب بيتمان لبيانات قياسية عادية من متتالية بمتغير واحد و من متتاليتين و قد تبين أن احتمال تقارب مقياس بيتمان للبيانات القياسية لا يعتمد على التوزيع الاحتمالي الذي تتبعه البيانات القياسية إذا كانت تلك البيانات تتبع لتوزيعات احتمالية متماثلة.

في هذه الأطروحة تم القيام بعمل تعميم للدراسات السابقة لحساب احتمال تقارب بيتمان لبيانات قياسية, حيث تبين بالنسبة للتوزيعات الاحتمالية المتماثلة أن احتمال تقارب مقياس بيتمان للبيانات القياسية لا يعتمد على التوزيع الاحتمالي الذي تتبعه البيانات القياسية. و كتوضيح لهذه النتائج عرضنا أمثلة لبيانات قياسية تتبع التوزيع الاحتمالي المنتظم و التوزيع الاحتمالي الأسي.